Distribution Theory
LECTURE 1

- Hörmander, Analysis of partial differential ops Vol. 1.
- Reed + Simon: Modern methods of Mathematical Physics Vol. 1
- Distribution theory, Friedlander.

§0: Motivation

Probably seen Dirac delta function
$$\int_{-\infty}^{\infty} \delta(x-x_0)\, \underset{\text{"nice"}}{f(x)}\,dx = f(x_0).$$

Can we define $\delta'(x-x_0)$? Try.
$$\int_{-\infty}^{\infty} \delta'(x-x_0) f(x)\,dx$$
$$= \lim_{h\to 0} \int_{-\infty}^{\infty}\left[\frac{\delta(x-x_0+h)-\delta(x-x_0)}{h}\right] f(x)\,dx.$$
$$= \lim_{h\to 0} \frac{1}{h}\left[f(x_0-h) - f(x_0)\right] = -f'(x_0).$$

i.e. $\int_{-\infty}^{\infty} \delta'(x-x_0) f(x)\,dx = -\int_{-\infty}^{\infty} \delta(x-x_0) f'(x)\,dx.$

Int-by-parts? Make rigorous using distribution theory.

Fourier transform of poly^{n}?

If $f\in L^1(\mathbb{R})$ $\left[\int_{\mathbb{R}} |f|\,dx<\infty\right]$
$$\hat f(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx.$$

How might we take F.T. of $f(x)=x^n$?
Might recall identity
$$\delta(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,dx$$

Might then get
$$\hat f(\lambda) = \int_{-\infty}^{\infty} x^n e^{-i\lambda x}\,dx.$$
$$= \left(i\frac{\partial}{\partial\lambda}\right)^n \int_{-\infty}^{\infty} e^{-i\lambda x}\,dx$$
$$= i^n\cdot 2\pi\cdot \delta^{(n)}(\lambda)$$

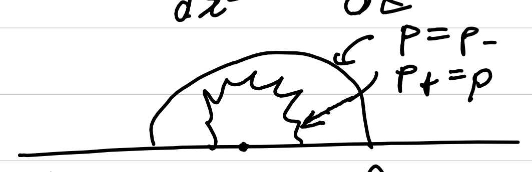
Recall Parseval's theorem
$$\int_{-\infty}^{\infty} \hat g(\lambda) f(\lambda)\,d\lambda = \int_{-\infty}^{\infty} g(x)\hat f(x)\,dx$$

DEFINE F.T. of $g(x)=x$ to be the f^n $\lambda \mapsto \hat x(\lambda)$ s.t.
$$\int_{-\infty}^{\infty} \hat x(\lambda) f(\lambda)\,d\lambda = \int_{-\infty}^{\infty} x \hat f(x)\,dx$$
for all NICE f^{ns} f.

Can make rigorous using distributions.

Discontinuous Sol^{n}s to PDEs

From linear acoustics, air pressure $p=p(x,t)$ satisfies wave eq^n:
$$\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} = 0. \quad (*)$$



Introduce nice $f=f(x,t)$, $f\in C_c^\infty(\mathbb{R}^2)$

$$(*) \Rightarrow \iint (p_{xx}-p_{tt}) f(x,t)\,dx\,dt = 0.$$
$$\Rightarrow \iint (f_{xx}-f_{tt})\, p(x,t)\,dx\,dt = 0$$

We say that $p=p(x,t)$ is a <u>weak</u> sol^n to $(*)$ if
$$\iint [f_{xx}-f_{tt}]\, p(x,t)\,dx\,dt = 0$$
$\forall f\in C_c^\infty(\mathbb{R}^2)$

[Try $p = H(x-t)$
$\quad$ ↑ Heaviside]

In each case, to extend a def^n to a larger domain of applicability, we had to introduce a space of "nice" functions. This is the theme of distribution theory: functions get replaced by linear maps on some auxillary space of test functions V. A distribution is a linear map $u: V\to \mathbb{C}$. I.e. we study the [topological] dual to V. Let $\langle\ ,\ \rangle$ denote the pairing between V and V^*, i.e. for $u\in V^*$, $f,g\in V$, $\alpha,\beta\in\mathbb{C}$, $\langle u, \alpha f+\beta g\rangle = \alpha\langle u,f\rangle + \beta\langle u,g\rangle$. The topological dual V^* consists of linear $u: V\to\mathbb{C}$ such that
$$f_n \xrightarrow{V} f \to \langle u, f_n\rangle \xrightarrow{\mathbb{C}} \langle u,f\rangle.$$

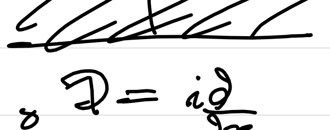
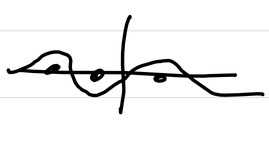
E.g. $V=C_c^\infty(\mathbb{R})$, equipped with the topology of local uniform convergence.
[i.e. $f_n\xrightarrow{V} f$ if $\forall$ compact $K\subset\mathbb{R}$ for all $n\geq 0$, $\sup_K \left|\left(\frac{d}{dx}\right)^n (f_n-f)\right| \to 0$.

$\delta_{x_0}: V\to\mathbb{C}$, $\langle \delta_{x_0}, f\rangle = f(x_0)$, $f\in V$.

Note, $\langle \delta_{x_0}, f_n\rangle \to \langle \delta_{x_0}, f\rangle$ if $f_n\to f$ in V.

$V = C_c^\infty(X)$, $X\subset\mathbb{R}^n$ open.
$f_n\xrightarrow{V} f$ $\quad$ Cinductive limit topology?

$g\in C_c^\infty(\mathbb{R})$
$$f\longmapsto \int f(x) g(x)\,dx, \quad f\in C_c^\infty(\mathbb{R})$$

[$u_{xx}+u_{yy}=0$ 
$P(D)u = f$ $\quad$ $D = i\frac{\partial}{\partial x}$ 
$P(\lambda)\hat u = \hat f$, $\quad \hat u = \frac{\hat f(\lambda)}{P(\lambda)}$

Distribution Theory
Lecture 2

§1: Distributions

§1.1: Notation + Preliminaries

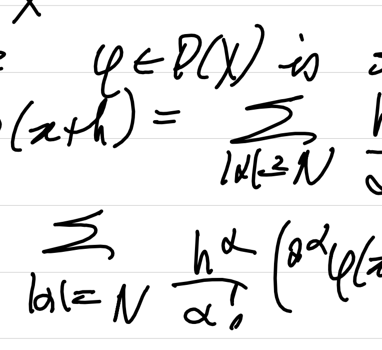
X, Y open subsets of $\mathbb{R}^n$.
K, compact
Integrals over X, $\mathbb{R}^n$ written
$\int_X [\cdot]\,dx$, $\int [\cdot]\,dx$

§1.2: Distributions + Test functions:

First space of test functions
Def^n 1.1

The space $\mathcal{D}(X)$ consists of smooth functions $\varphi : X \to \mathbb{C}$, which have compact support. We say a sequence $\{\varphi_n\}$ in $\mathcal{D}(X)$ tends to zero in $\mathcal{D}(X)$
$$\left[\varphi_n \xrightarrow{\mathcal{D}(X)} 0\right]$$
if: $\exists$ compact $K \subset X$ such that $\text{supp}(\varphi_n) \subset K$ and $\sup|\partial^\alpha \varphi_n| \to 0$ for each multi-index α.

Functions in $\mathcal{D}(X)$ have nice properties, e.g. if $\varphi \in \mathcal{D}(X)$ then $\varphi \equiv 0$ before you reach ∂X.



$A = \mathbb{R}^n \setminus X$ = closed
∂X, $\text{dist}(A, \text{supp}\,\varphi) \geq \delta > 0$.
closed compact.

This means that integration-by-parts is easy
$$\int_X \varphi\, \partial^\alpha \psi\, dx = (-1)^{|\alpha|} \int_X \partial^\alpha \varphi\, \psi\, dx$$
Since $\varphi \in \mathcal{D}(X)$ is smooth
$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + \underbrace{R_N(x,h)}_{o(|h|^N)}$$
$$\sum_{|\alpha| = N} \frac{h^\alpha}{\alpha!} \left(\partial^\alpha \varphi(x) - \partial^\alpha \varphi(\xi)\right) \quad \text{uniformly in } x.$$
$\xi \in B_h(x)$.
$\partial^\alpha \varphi$ smooth, compactly supported $\Rightarrow$ uniformly continuous $\Longrightarrow$ error $\left|\frac{R_N(x,h)}{|h|^N}\right| \to 0$, $h \to 0$ uniformly in h.

Def^n: 1.2
A linear map $u : \mathcal{D}(X) \to \mathbb{C}$ is called a <u>distribution</u> if: $\forall$ compact $K \subset X$, $\exists\, C, N > 0$ such that
$$|\langle u, \varphi \rangle| \leq C \cdot \sum_{|\alpha| \leq N} \sup|\partial^\alpha \varphi|$$
$u(\varphi)$
for all $\varphi \in \mathcal{D}(X)$ with $\text{supp}(\varphi) \subset K$. Space of all such maps denoted by $\mathcal{D}'(X)$, "distributions on X." If the same N can be used in $(*)$ for all compact $K \subset X$, say least such N is the <u>order</u> of u, written $\text{ord}(u)$.

[Note $N \equiv N_K$, $C \equiv C_K$]

For $x_0 \in X$, define
$$\langle \delta_{x_0}, \varphi \rangle = \varphi(x_0), \quad \varphi \in \mathcal{D}(X).$$

Then $\delta_{x_0} : \mathcal{D}(X) \to \mathbb{C}$ is linear and
$$|\langle \delta_{x_0}, \varphi \rangle| = |\varphi(x_0)| \leq \sup|\varphi|$$

So $C = 1$, $N = 0$ in $(*)$. See that $\text{ord}(\delta_{x_0}) = 0$.

For $\{f_\alpha\}$ in $C(X)$, define $T : \mathcal{D}(X) \to \mathbb{C}$
$$\langle T, \varphi \rangle = \sum_{|\alpha| \leq M} \int_X f_\alpha \partial^\alpha \varphi\, dx$$
Take $\varphi \in \mathcal{D}(X)$, $\text{supp}(\varphi) \subset K$. Then
$$|\langle T, \varphi \rangle| \leq \sum_{|\alpha| \leq M} \int_K |f_\alpha| \cdot |\partial^\alpha \varphi|\, dx$$
$$\leq \left(\max_{|\alpha| \leq M} \int_K |f_\alpha|\, dx\right) \cdot \sum_{|\alpha| \leq M} \sup|\partial^\alpha \varphi|.$$
So $(*)$ holds with $C = \max_\alpha \int_K |f_\alpha|\, dx$, $N = M$
So $T \in \mathcal{D}'(X)$.
Note this estimate would hold if the $\{f_\alpha\}$ were just locally integrable, written $f_\alpha \in L^1_{loc}(X)$. [$\forall$ compact $K \subset X$, $\int_K |f_\alpha|\, dx < \infty$].

[ABUSE OF NOTATION:
if $f \in L^1_{loc}(X)$, can define
$$T_f : \mathcal{D}(X) \to \mathbb{C}, \quad \langle T_f, \varphi \rangle = \int_X f \varphi\, dx$$
[previous example, $M = 0$]. We simply write $T_f \equiv f$.]

Lemma 1.1:
Let $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then a linear map $u : \mathcal{D}(X) \to \mathbb{C}$ belongs to $\mathcal{D}'(X)$ iff $\lim_{m \to \infty} \langle u, \varphi_m \rangle = 0$ for all such sequences.

[$\langle u, \varphi_m \rangle \to \langle u, \varphi \rangle$, $\varphi_m \to \varphi$
$\varphi_m \to \varphi \Leftrightarrow \varphi_m - \varphi \to 0$]

Proof:
($\Rightarrow$) Suppose $u \in \mathcal{D}'(X)$ and $\varphi_m \to 0$ in $\mathcal{D}(X)$. Then $\text{supp}(\varphi_m) \subset K$, and by semi-norm estimate, $\exists\, C, N > 0$:
$$|\langle u, \varphi_m \rangle| \leq C \cdot \sum_{|\alpha| \leq N} \sup|\partial^\alpha \varphi_m| \to 0.$$
($\Leftarrow$) Suppose not, i.e. $u : \mathcal{D}(X) \to \mathbb{C}$ linear and $\varphi_m \to 0$ in $\mathcal{D}(X) \Rightarrow \langle u, \varphi_m \rangle \to 0$, but no estimate of form $(*)$ holds. I.e., there exists a compact set $K \subset X$, such that for all choices of C, N, $(*)$ fails on some test function with $\text{supp}(\varphi) \subset K$. In particular if we take $C = N = m$, there must exist some $\varphi_m \in \mathcal{D}(X)$ with $\text{supp}(\varphi_m) \subset K$ and $|\langle u, \varphi_m \rangle| > m \cdot \sum_{|\alpha| \leq m} \sup|\partial^\alpha \varphi_m|$.

Can replace φ_m with $\varphi_m / \langle u, \varphi_m \rangle$.
So $\langle u, \varphi_m \rangle = 1$ wlog. I.e.
$$1 > m \cdot \sum_{|\alpha| \leq m} \sup|\partial^\alpha \varphi_m|$$
$$\Rightarrow \sup|\partial^\alpha \varphi_m| < 1/m, \quad m \geq |\alpha|.$$

But the $\varphi_m \to 0$ in $\mathcal{D}(X)$, a contradiction since $\langle u, \varphi_m \rangle = 1 \not\to 0$. ∎

§1.3: Limits in $\mathcal{D}'(X)$

Often have sequence $\{u_m\}$ in $\mathcal{D}'(X)$. If there is some $u \in \mathcal{D}'(X)$ such that
$$\langle u_m, \varphi \rangle \to \langle u, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$
say $u_m \xrightarrow{\mathcal{D}'(X)} u$.

Theorem 1.1 [NON-EXAMINABLE]

If $\{u_m\}$ sequence in $\mathcal{D}'(X)$ and $\langle u, \varphi \rangle := \lim_{m \to \infty} \langle u_m, \varphi \rangle$ exists for all $\varphi \in \mathcal{D}(X)$, then $u \in \mathcal{D}'(X)$.

Take $u_m \in \mathcal{D}'(X)$ defined by
$$\langle u_m, x \rangle = \int \sin(mx) \varphi(x)\, dx$$

[$u_m = \sin mx$].

On IBP $|\langle u_m, \varphi \rangle| = \left|\frac{1}{m} \int \cos(mx) \cdot \varphi'(x)\, dx\right|$
$\to 0$ as $m \to \infty$.
I.e. $\sin(mx) \to 0$ in $\mathcal{D}'(X)$.

LECTURE 3

§1.4: Basic Operations:

§1.4.1: Differentiation + Multiplication by smooth f^ns.

For $u \in C^\infty(X) \subseteq L^1_{loc}(X)$, $\partial^\alpha u \in D'(X)$

$$\langle \partial^\alpha u, \varphi \rangle = \int_X \varphi \partial^\alpha u \, dx, \ \varphi \in D(X).$$
$$= (-1)^{|\alpha|} \int_X \partial^\alpha \varphi \, u \, dx$$

Leads to:

Def^n 1.3:

For $u \in D(X)$, $f \in C^\infty(X)$, define
$$\langle \partial^\alpha (fu), \varphi \rangle := (-1)^{|\alpha|} \langle u, f \partial^\alpha \varphi \rangle$$
for $\varphi \in D(X)$. Call $\partial^\alpha u$ the <u>distributional</u> <u>derivatives</u> of u.

⌈Note $\partial^\alpha (fu) \in D'(X)$⌋

For δ_x have:
$$\langle \partial^\alpha \delta_x, \varphi \rangle = (-1)^{|\alpha|} \langle \delta_x, \partial^\alpha \varphi \rangle$$
$$= (-1)^{|\alpha|} \partial^\alpha \varphi(x).$$

Define Heaviside f^n:
$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

$H \in L^1_{loc}(\mathbb{R})$.
$$\langle H', \varphi \rangle := - \langle H, \varphi' \rangle$$
$$= - \int_0^\infty \varphi'(x) dx = \varphi(0).$$
$$= \langle \delta_0, \varphi \rangle.$$

So, $H' = \delta_0$. Generally, say $u = v$ in $D'(X)$ if $\langle u, \varphi \rangle = \langle v, \varphi \rangle \ \forall \varphi \in D(X)$.

Lemma 1.2:

If $u \in D'(\mathbb{R})$ and $u' = 0$ in $D'(X)$, then $u = \text{const}$.

Proof:

Fix $\theta \in D(\mathbb{R})$, $\langle 1, \theta \rangle = \int \theta dx = 1$.

For $\varphi \in D(\mathbb{R})$ write
$$\varphi = (\varphi - \langle 1, \varphi \rangle \theta) + \langle 1, \varphi \rangle \theta$$
$$= \varphi_A + \varphi_B$$

Note that $\langle 1, \varphi_A \rangle = \int \varphi_A dx = 0$.

So we have
$$\psi_A(x) = \int_{-\infty}^x \varphi_A(t) dt$$

So $\psi_A \in D(\mathbb{R})$, $\psi_A' = \varphi_A$. So
$$\langle u, \varphi \rangle = \langle u, \varphi_A \rangle + \langle u, \varphi_B \rangle$$
$$= \langle u, \psi_A' \rangle + \langle 1, \varphi \rangle \cdot \underbrace{\langle u, \theta \rangle}_{= \text{constant} = c.}$$
$$= - \cancel{\langle u', \psi_A \rangle}^{0} + \langle c, \varphi \rangle$$

So $u = \text{const}$ in $D'(\mathbb{R})$. □

§1.4.2: Translation + Reflection

If $\varphi \in D(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$, define reflection + translation by
$$\check{\varphi}(x) = \varphi(-x), \quad (\tau_h \varphi)(x) = \varphi(x-h).$$

Def^n 1.4:

For $u \in D'(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$ define
$$\langle \check{u}, \varphi \rangle = \langle u, \check{\varphi} \rangle.$$
$$\langle \tau_h u, \varphi \rangle = \langle u, \tau_{-h} \varphi \rangle$$
for $\varphi \in D(\mathbb{R}^n)$.

Lemma 1.3:

for $u \in D'(\mathbb{R}^n)$ define
$$V_h = \frac{\tau_{-h} u - u}{|h|}$$
If $\frac{h}{|h|} \to m \in S^{n-1}$ as $|h| \to 0$, then
$$V_h \to m \cdot \partial u \text{ in } D'(\mathbb{R}^n).$$

Proof:

For $\varphi \in D(\mathbb{R}^n)$, by def^n of V_h
$$\langle V_h, \varphi \rangle = \left\langle u, \frac{\tau_h \varphi - \varphi}{|h|} \right\rangle$$
By Taylor's th^m:
$$(\tau_h \varphi - \varphi)(x) = \varphi(x-h) - \varphi(x)$$
$$= - \sum_i h_i \frac{\partial \varphi}{\partial x_i} + R_1(x, h)$$

⌈DAMPT Part III examples for details⌋

Know that $R_1 = o(|h|)$ in $D(\mathbb{R}^n)$, so by sequential continuity [Lemma 1.1]
$$\langle v_h, \varphi \rangle = - \sum_i \frac{h_i}{|h|} \langle u, \frac{\partial \varphi}{\partial x_i} \rangle + o(1)$$
$$= \left\langle \sum \frac{h_i}{|h|} \frac{\partial u}{\partial x_i}, \varphi \right\rangle + o(1).$$
$$\to \langle m \cdot \partial u, \varphi \rangle \text{ as } |h| \to 0$$ ■

§1.4.3: Convolution between $D'(\mathbb{R}^n)$ and $D(\mathbb{R}^n)$

For $\varphi \in D(\mathbb{R}^n)$, have
$$(\tau_x \check{\varphi})(y) = \check{\varphi}(y-x) = \varphi(x-y)$$

If $u \in C^\infty(\mathbb{R}^n)$ define convolution with $\varphi \in D(\mathbb{R}^n)$:
$$u * \varphi(x) = \int u(x-y) \varphi(y) dy$$
$$= \int \varphi(x-y) u(y) dy$$
$$= \langle u, \tau_x \check{\varphi} \rangle$$

Def^n 1.5:

For $u \in D'(\mathbb{R}^n)$, $\varphi \in D(\mathbb{R}^n)$ define:
$$u * \varphi(x) = \langle u, \tau_x \check{\varphi} \rangle$$

How regular is $u * \varphi(x)$?

Lemma 1.4:

For $\phi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, write $\Phi_x(y) = \varphi(x, y)$. If for each $x \in \mathbb{R}^n$, exists neighbourhood $N_x \subseteq \mathbb{R}^n$ and compact $K \subset \mathbb{R}^n$ such that
$$\text{supp}\left(\phi|_{N_x \times \mathbb{R}^n}\right) \subseteq N_x \times K$$
then $\partial_x^\alpha \langle u, \Phi_x \rangle = \langle u, \partial_x^\alpha \Phi \rangle$ for $u \in D'(\mathbb{R}^n)$.

⌈Think of $y \mapsto \Phi_x(y)$ as family of test functions⌋

Proof: From def^n and Taylor's th^m
$$\Phi_{x+h}(y) - \Phi_x(y)$$
$$= \sum_i h_i \frac{\partial \Phi}{\partial x_i}(x, y) + R_1(x, y, h).$$

For $|h|$ sufficiently small, $x + h \in N_x$ so $\text{supp}(R_1(x, \cdot, h)) \subseteq K$, also have $\sup_y |\partial_y^\alpha R_1(x, y, h)| = o(|h|)$. So
$$R_1(x, \cdot, h) = o(|h|) \text{ in } D(\mathbb{R}^n).$$
By sequential continuity:
$$\langle u, \Phi_{x+h} \rangle - \langle u, \Phi_x \rangle$$
$$= \sum h_i \langle u, \frac{\partial}{\partial x_i} \Phi_x \rangle + o(|h|).$$

So $\frac{\partial}{\partial x_i} \langle u, \Phi_x \rangle = \langle u, \frac{\partial}{\partial x_i} \Phi_x \rangle$, result follows by induction.

Corollary 1.1:

If $u \in D'(\mathbb{R}^n)$ and $\varphi \in D(\mathbb{R}^n)$ then $u * \varphi \in C^\infty(\mathbb{R}^n)$ and
$$\partial^\alpha (u * \varphi) = u * \partial^\alpha \varphi$$

⌈$(u * \varphi)(x) = \langle u, \tau_x \check{\varphi} \rangle$ so $\Phi_x = \tau_x \check{\varphi}$ in previous lemma⌋

LECTURE 4

§1.5: Density of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{D}'(\mathbb{R}^n)$

Can use previous result to prove important theorem. Need:

Lemma 1.5:
If $u \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$
then $(u * \varphi) * \psi = u * (\varphi * \psi)$

Proof:
Fix $x \in \mathbb{R}^n$.

$$(u*\varphi)*\psi(x) = \int_{\mathbb{R}^n} (u*\varphi)(x-y)\psi(y)\,dy$$

$$= \int \langle u(z), \varphi(x-y-z)\rangle \psi(y)\,dy \quad \longleftarrow \text{ABUSE}$$

$$= \int \langle u(z), \varphi(x-y-z)\cdot\psi(y)\rangle\,dy$$

$$= \lim_{h\to 0} \sum_{m\in\mathbb{Z}^n} \langle u(z), \varphi(x-z-hm)\psi(hm)\rangle h^n$$

$$= \lim_{h\to 0} \langle u(z), \underbrace{\sum_{m\in\mathbb{Z}^n} \varphi(x-z-hm)\psi(hm)h^n}_{\to\ \varphi*\psi(x-z) \text{ in } \mathcal{D}(\mathbb{R}^n)} \rangle \quad \text{Finite sum } \forall h>0.$$

(+)

$$= \langle u(z), \varphi*\psi(x-z)\rangle \quad \text{(By sequential continuity)}$$

$$= u*(\varphi*\psi)(x).$$

(+) Define, for $|h|\le 1$, family of functions $\{F_h\}$

$$z \longmapsto \sum_{m\in\mathbb{Z}^n} \varphi(x-z-hm)\psi(hm)\cdot h^n$$

Straightforward to show that $\mathrm{supp}(F_h)$ lies in some fixed, compact $K\subset\mathbb{R}^n$.
Also, F_h are smooth. Note that for each α, $\sup_z |\partial^\alpha F_h(z)| \le M_\alpha$
So for each α, $z \mapsto \partial^\alpha F_h(z)$ is uniformly bounded and equicontinuous:

$$|\partial^\alpha F_h(x) - \partial^\alpha F_h(y)| = \left|\int_0^1 \frac{d}{dt}\partial^\alpha F_h(tx+(1-t)y)\,dt\right|$$

$$= \left|\int_0^1 (x-y)\cdot\nabla\partial^\alpha F_h(tx+(1-t)y)\,dt\right|$$

$$\lesssim_\alpha |x-y|, \qquad (A\lesssim B \Rightarrow \exists C>0 : A\le C\cdot B).$$

Applying Arzela-Ascoli and diagonal argument, get sequence $\{h_k\}$ s.t.

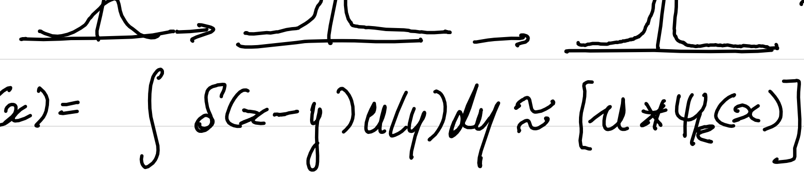
$$\sup_z |\partial^\alpha F_{h_k}(z) - \partial^\alpha \varphi*\psi(x-z)| \to 0$$

for each α

NON-EXAMINABLE

Theorem 1.2:
For $u \in \mathcal{D}'(\mathbb{R}^n)$, $\exists \{\varphi_k\}$ in $\mathcal{D}(\mathbb{R}^n)$ s.t. $\varphi_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$.
[$u_k \to u$ in $\mathcal{D}'(\mathbb{R}^n)$ if $\langle u_k, \theta\rangle \to \langle u, \theta\rangle$ $\forall\theta\in\mathcal{D}(\mathbb{R}^n)$].

Proof:
$\psi_k \in \mathcal{D}(\mathbb{R}^n)$, $\int \psi_k\,dx = 1$, $\mathrm{supp}(\psi_k) \to \{0\}$.



"$\psi_k \overset{?}{\to} \delta_0$".

$$u(x) = \int \delta(x-y)u(y)\,dy \approx [u*\psi_k(x)]\cdot\chi_k(x)$$

$$\chi = \begin{cases} 1, & |x|<1 \\ 0, & |x|>2 \end{cases}, \quad \chi_k(x) = \chi(x/k),\ \chi\in C_c^\infty(\mathbb{R}^n)$$

Fix $\psi\in\mathcal{D}(\mathbb{R}^n)$ with $\int\psi\,dx = 1$, set

$$\psi_k(x) = k^n\psi(kx) \quad \left[\int\psi_k\,dx = 1\right]$$

Fix $\chi\in\mathcal{D}(\mathbb{R}^n)$ with $\chi = 1$, on $|x|<1$ and $\chi = 0$ on $|x|\ge 2$. For $u\in\mathcal{D}'(\mathbb{R}^n)$, and arbitrary $\theta\in\mathcal{D}(\mathbb{R}^n)$, consider $\langle\varphi_k,\theta\rangle$ where
$\varphi_k = (u*\psi_k)\chi_k$, $\chi_k(x) = \chi(\frac{x}{k})$.

$$\langle\varphi_k,\theta\rangle = \langle u*\psi_k, \chi_k\theta\rangle$$

$$= (u*\psi_k)*(\chi_k\theta)^\vee(0) \quad [\langle v,f\rangle = (v*\check{f})(0)]$$

$$= u*(\psi_k*(\chi_k\theta)^\vee)(0)$$

[By Lemma 1.5]

$$\psi_k*(\chi_k\theta)^\vee(x) \qquad y' = k(x-y) \Leftrightarrow y = x - y'/k$$

$$= \int k^n\cdot\psi(k(x-y))\cdot\chi\left(-\frac{y}{k}\right)\theta(-y)\,dy$$

$$= \int\psi(y)\cdot\chi\left(\frac{y}{k^2}-\frac{x}{k}\right)\theta\left(\frac{y}{k}-x\right)dy$$

$$= \theta(-x) + R_k(-x), \text{ where} \qquad \int\psi\,dy = 1.$$

$$R_k(x) = \int\psi(y)\left[\chi\left(\frac{y}{k^2}+\frac{x}{k}\right)\theta\left(\frac{y}{k}+x\right) - \theta(x)\right]dy$$

So: $\langle\varphi_k,\theta\rangle = u*\check{\theta}(0) + u*\check{R}_k(0)$
$= \langle u,\theta\rangle + \langle u, R_k\rangle$

Straightforward to show that $R_k\to 0$ in $\mathcal{D}(\mathbb{R}^n)$ [exercise], sequential continuity implies that $\langle\varphi_k,\theta\rangle \to \langle u,\theta\rangle$ ■

§2: Distributions of Compact Support

Let $Y\subset X$ be open. We say $u\in\mathcal{D}'(X)$ vanishes on Y if $\langle u,\varphi\rangle = 0$ $\forall\varphi\in\mathcal{D}(Y)$.

Def^n 2.1:
For $u\in\mathcal{D}'(X)$ define support of u by

$$\mathrm{supp}(u) = X \setminus \left\{ \bigcup_{\substack{Y\subset X\\ \text{open}}} \{u \text{ vanishes on } Y\} \right\}$$

Eg. for $\delta_x\in\mathcal{D}'(\mathbb{R}^n)$, $\mathrm{supp}(\delta_x) = \{x\}$.

If $u\in\mathcal{D}'(X)$ vanishes on collection $\{U_i\}$ of open sets, then u vanishes on their union. Indeed, suppose $\mathrm{supp}(\varphi)\subset\bigcup_i U_i$. By compactness $\exists$ $\{U_i\}_{i=1}^N$ such that $\mathrm{supp}(\varphi)\subset\bigcup_{i=1}^N U_i$.
Invoke partition of unity $\{\psi_i\}_{i=1}^N$ subordinate to $\{U_i\}_{i=1}^N$, i.e. $\mathrm{supp}\,\psi_i\subset U_i$ and $\sum_{i=1}^N\psi_i = 1$.
Then,

$$\langle u,\varphi\rangle = \langle u, \sum_{i=1}^N\psi_i\varphi\rangle$$

$$= \sum_{i=1}^N\langle u,\psi_i\varphi\rangle = 0$$

$\hookrightarrow \mathrm{supp}(\psi_i\varphi)\subset U_i$.

Corollary:
$\mathrm{supp}(u)$ = Complement of largest open set on which u vanishes

LECTURE 5

§ 2.1: More test functions + Distribution

Definition 2.2:
Define $\mathcal{E}(X)$ to be the space of smooth functions $\varphi : X \to \mathbb{C}$. We say $\varphi_\mu \to 0$ in $\mathcal{E}(X)$ if, for each multi-index α, have
$\partial^\alpha \varphi_\mu \to 0$ locally uniformly.
[i.e., $\sup_K |\partial^\alpha \varphi_\mu| \to 0$ $\forall K \subset X$ compact].

Definition 2.3:
A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ if: $\exists$ compact $K \subset X$, constants $C, N \geq 0$ such that

$$|\langle u, \varphi \rangle| \leq C \cdot \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$$

for all $\varphi \in \mathcal{E}(X)$.
$\mathcal{E}'(X) = \{$distributions with compact support$\}$.
Note: $\mathrm{supp}(u) \subset K$, but $\mathrm{supp}(u) \neq K$ in general.
[Friedlander goes through a counter example].

Lemma 2.1:
A linear map $u : \mathcal{E}(X) \to \mathbb{C}$ belongs to $\mathcal{E}'(X)$ iff $\langle u, \varphi_m \rangle \to 0$ for all $\{\varphi_m\}$ s.t. $\varphi_m \to 0$ in $\mathcal{E}(X)$.

Proof: almost identical to analogous result for $\mathcal{D}'(X)$. C.F. compact exhaustion of X.
$(\Rightarrow)$: $\exists K \subset X$ compact & $C, N \geq 0$ s.t.

$$\forall \varphi \in \mathcal{E}(X): \quad |\langle u, \varphi \rangle| \leq C \cdot \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$$

Thus, $\varphi_m \to 0$ in $\mathcal{E}(X) \Rightarrow \sup_K |\partial^\alpha \varphi_m| \to 0$ $\forall \alpha$, $m \to \infty$
$\Rightarrow |\langle u, \varphi_m \rangle| \to 0$ as required.
$(\Leftarrow)$: Suppose $u \notin \mathcal{E}'(X) \Rightarrow \forall K \subset X$ compact, $\forall C, N \geq 0$, $\exists \varphi \in \mathcal{E}(X)$ s.t. $|\langle u, \varphi \rangle| > C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi|$
Let $(K_n)_n$ be a compact exhaustion of X. Then, $\exists (\varphi_n)$ s.t. $|\langle u, \varphi_n \rangle| > n \cdot \sum_{|\alpha| \leq n} \sup_{K_n} |\partial^\alpha \varphi_n|$ $\forall n \in \mathbb{N}$. Wlog suppose $\langle u, \varphi_n \rangle = 1$ $\forall n$

$$\Rightarrow \sum_{|\alpha| \leq n} \sup_{K_n} |\partial^\alpha \varphi_n| \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \quad \varphi_n \to 0 \text{ in } \mathcal{E}(X) \text{ but } \langle u, \varphi_n \rangle = 1 \ \forall n \ \lightning$$

Lemma 2.2:
If $u \in \mathcal{E}'(X)$, then $u|_{\mathcal{D}(X)}$ defines an element of $\mathcal{D}'(X)$ with compact support. Conversely, if $u \in \mathcal{D}'(X)$ has compact support, there exists a unique $\tilde{u} \in \mathcal{E}'(X)$ such that $\mathrm{supp}(u) = \mathrm{supp}(\tilde{u})$ and $\tilde{u}|_{\mathcal{D}(X)} = u$.

Proof: Note that $\mathcal{D}(X) \subset \mathcal{E}(X)$, so if $u \in \mathcal{E}'(X)$ then $u|_{\mathcal{D}(X)}$ well-defined. There exists compact $K \subset X$, constants $C, N \geq 0$ s.t.

$$|\langle u, \varphi \rangle| \leq C \cdot \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \quad \forall \varphi \in \mathcal{D}(X).$$

So $u|_{\mathcal{D}(X)} \in \mathcal{D}'(X)$ and $\mathrm{supp}(u) \subset K$.
If $u \in \mathcal{D}'(X)$ with compact support, fix $\rho \in \mathcal{D}(X)$ s.t. $\rho = 1$ on a nbhd of $\mathrm{supp}(u)$.
Define $\tilde{u} : \mathcal{E}(X) \to \mathbb{C}$ by
$\langle \tilde{u}, \varphi \rangle = \langle u, \rho\varphi \rangle$, $\varphi \in \mathcal{E}(X)$.
Note that $\mathrm{supp}(\rho\varphi) \subset \mathrm{supp}(\rho) \equiv K$. So since $u \in \mathcal{D}'(X)$, $\exists$ constants $c, N \geq 0$ s.t.

$$\begin{aligned}
|\langle \tilde{u}, \varphi \rangle| &= |\langle u, \rho\varphi \rangle| \\
&\leq C \cdot \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha (\rho\varphi)| \\
&\leq C' \cdot \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \varphi| \quad \text{(by Leibniz).}
\end{aligned}$$

So $\tilde{u} \in \mathcal{E}'(X)$. Suppose $\exists \tilde{v} \in \mathcal{E}'(X)$ with $\tilde{v}|_{\mathcal{D}(X)} = u = \tilde{u}|_{\mathcal{D}(X)}$ and $\mathrm{supp}(\tilde{v}) = \mathrm{supp}(\tilde{u}) = \mathrm{supp}(u)$.
With $\rho \in \mathcal{D}(X)$ as before,

$$\begin{aligned}
\langle \tilde{v}, \varphi \rangle &= \langle \tilde{v}, \rho\varphi \rangle + \langle \tilde{v}, (1-\rho)\varphi \rangle \\
&= \langle \tilde{u}, \rho\varphi \rangle + \langle \tilde{u}, (1-\rho)\varphi \rangle \\
&= \langle \tilde{u}, \varphi \rangle.
\end{aligned}$$

$= 0$ on nbhd of $\mathrm{supp}(u)$ $= \mathrm{supp}(\tilde{v})$

$\square$

§ 2.2: Convolution between $\mathcal{E}'(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$

For $\varphi \in \mathcal{E}(\mathbb{R}^n)$, $u \in \mathcal{E}'(\mathbb{R}^n)$ define convolution as before

$$u * \varphi(x) = \langle u, \tau_x \check{\varphi} \rangle$$

Find $u * \varphi \in \mathcal{E}(\mathbb{R}^n)$. Note that $u * \varphi = 0$ unless $(x-y) \in \mathrm{supp}(\varphi)$ for some $y \in \mathrm{supp}(u)$ i.e. $\mathrm{supp}(u * \varphi) \subset \mathrm{supp}(\varphi) + \mathrm{supp}(u)$. In particular, if $u \in \mathcal{E}'(\mathbb{R}^n)$, $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then $u * \varphi \in \mathcal{D}(\mathbb{R}^n)$.

Def^n 2.4:
Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$, at least one of which has compact support. Then define

$$(u * v) * \varphi := u * (v * \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n).$$

[then $u * v \in \mathcal{D}'(\mathbb{R}^n)$, see ES 2.]

Lemma 2.3:
For u, v as in Def^n 2.4, $u * v = v * u$.

Proof:
Recall Lemma 1.5, if $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ then $(u * \varphi) * \psi = u * (\varphi * \psi)$.
Same holds if $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{E}(\mathbb{R}^n)$ with at least one of $\mathrm{supp}(\varphi), \mathrm{supp}(\psi)$ compact.
For $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$

$$\begin{aligned}
(u * v) * (\varphi * \psi) &\longleftarrow (\dagger) \\
&= u * [v * (\varphi * \psi)] \\
&= u * [(v * \varphi) * \psi] \\
&= u * [\psi * (v * \varphi)] \\
&= (u * \psi) * (v * \varphi).
\end{aligned}$$

Use $\varphi * \psi = \psi * \varphi$, then

$$\begin{aligned}
(v * u) * (\varphi * \psi) &= (v * \varphi) * (u * \psi) \\
&= (u * \psi) * (v * \varphi) \\
&= (\dagger)
\end{aligned}$$

So if $E = u * v - v * u$, then $E * (\varphi * \psi) = 0$ $\forall \varphi, \psi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow (E * \varphi) * \psi = 0$ $\forall \varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$
$\Rightarrow (E * \varphi) = 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n)$
$\Rightarrow E = 0$ in $\mathcal{D}'(\mathbb{R}^n)$, i.e.
$u * v = v * u$ [$\langle u, \varphi \rangle = u * \check{\varphi}(0)$] $\square$

So for any $u \in \mathcal{D}'(\mathbb{R}^n)$ have

$$\delta_0 * u = u * \delta_0 = u$$

since, for $\psi \in \mathcal{D}(\mathbb{R}^n)$

$$(u * \delta_0) * \psi = u * (\delta_0 * \psi) = u * \psi$$

and

$$\begin{aligned}
(\delta_0 * \psi)(x) &= \langle \delta_0, \tau_x \check{\psi} \rangle \\
&= (\tau_x \check{\psi})(0) \\
&= \check{\psi}(-x) = \psi(x).
\end{aligned}$$

LECTURE 6

§3: Tempered Distributions + Fourier Analysis

§3.1: More test functions & distributions

Definition 3.1

The Schwarz space, written $S(\mathbb{R}^n)$ consists of smooth $\varphi: \mathbb{R}^n \to \mathbb{C}$ s.t.

$$\|\varphi\|_{\alpha,\beta} := \sup |x^\alpha D^\beta \varphi| < \infty$$

for all multi-indices, α, β. Say $\varphi_n \xrightarrow{S} 0$ if $\|\varphi_n\|_{\alpha,\beta} \to 0$ $\forall \alpha, \beta$.

[Functions of rapid decay]

Def^n 3.2:

A linear map $u: S(\mathbb{R}^n) \to \mathbb{C}$ belongs to $S'(\mathbb{R}^n)$, space of tempered distributions if $\exists C, N \geq 0$ s.t.

$$|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|,|\beta| \leq N} \|\varphi\|_{\alpha,\beta}$$

$\forall \varphi \in S(\mathbb{R}^n)$.

[Can show $u: S(\mathbb{R}^n) \to \mathbb{C}$ (linear) belongs to $S'(\mathbb{R}^n) \Leftrightarrow \langle u, \varphi_m \rangle \to 0$ $\forall$ $\varphi_m \xrightarrow{S} 0$]

Standard Schwarz f^n

$$\varphi(x) = e^{-|x|^2}$$

Note that

$$\mathcal{D}(\mathbb{R}^n) \subset S(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$$

[$\varphi_m \to 0$ in $\mathcal{D} \Rightarrow \varphi_m \to$ in $S \Rightarrow \varphi_m \to 0$ in $\mathcal{E}$]

and $\mathcal{E}'(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$.

S, S' are the Goldilocks pair if we want to do Fourier Analysis.

§3.2: Fourier Transform on $S(\mathbb{R}^n)$

Def^n 3.3:

For $f \in L^1(\mathbb{R}^n)$ define Fourier Transform

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} e^{-i\lambda x} f(x)\,dx, \quad \lambda \in \mathbb{R}^n.$$

use $\mathcal{F}$ to denote linear map $\mathcal{F}: f \mapsto \hat{f}$.

Note that $S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$, since for $\varphi \in S(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\varphi|\,dx = \int_{\mathbb{R}^n} (1+|x|)^{-N} (1+|x|)^N |\varphi|\,dx$$
$$\leq C \sum_{|\alpha| \leq N} \|\varphi\|_{\alpha,0} \cdot \int_{\mathbb{R}^n} (1+|x|)^{-N} dx < \infty$$

for $N \geq n+1$.

Lemma 3.1:

If $f \in L^1(\mathbb{R}^n) \Rightarrow \hat{f} \in C(\mathbb{R}^n)$.

Proof:

Suppose $\lambda_k \to \lambda$ in $\mathbb{R}^n$. Then

$$\lim_{k\to\infty} \hat{f}(\lambda_k) = \lim_{k\to\infty} \int e^{-i\lambda_k \cdot x} \underbrace{f(x)\,dx}_{g_k(x)} \stackrel{=}{}$$

$|g_k| \leq |f|$, $f \in L^1(\mathbb{R}^n)$

$$\stackrel{=}{} \int e^{-i\lambda\cdot x} f(x)\,dx = \hat{f}(\lambda) \qquad \square$$

IMPORTANT: Fourier transform interchanges smoothness and decay.

Lemma 3.2

For $\varphi \in S(\mathbb{R}^n)$ have

$$(D^\alpha \varphi)^\wedge(\lambda) = \lambda^\alpha \hat{\varphi}(\lambda) \qquad D \equiv -i\partial$$
$$(x^\beta \varphi)^\wedge(\lambda) = (-D_\lambda)^\beta \hat{\varphi}(\lambda) \qquad \frac{D}{Dx_1} \equiv -i\frac{\partial}{\partial x_1}$$

[Note REGULARITY $\longleftrightarrow$ DECAY]

Proof:

By IBP:

$$(D^\alpha\varphi)^\wedge(\lambda) = \int e^{-i\lambda\cdot x} D^\alpha \varphi\,dx$$
$$= (-1)^{|\alpha|} \int \varphi D_x^\alpha [e^{-i\lambda\cdot x}]\,dx$$
$$= \cancel{(-1)^{|\alpha|}} \int \cancel{(-1)^{|\alpha|}} \varphi \lambda^\alpha e^{-i\lambda\cdot x}\,dx$$
$$= \lambda^\alpha \hat{\varphi}(\lambda).$$

$$(-D_\lambda)^\beta \hat{\varphi}(\lambda) \quad (-D_\lambda)^\beta \int e^{-i\lambda\cdot x} \varphi(x)\,dx$$
$$\stackrel{DCT}{=} \int x^\beta e^{-i\lambda\cdot x} \varphi(x)\,dx$$
$$= (x^\beta \varphi)^\wedge(\lambda) \qquad \square$$

[Note Lemmas 3.1, 3.2 $\Rightarrow \mathcal{F}: S(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$]

Might have seen "Fourier inversion"

$$\varphi(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} \hat{\varphi}(\lambda)\,d\lambda.$$

Theorem 3.1:

The Fourier Transform is a continuous isomorphism on $S(\mathbb{R}^n)$ is bijective and $\varphi_m \xrightarrow{S} 0 \Leftrightarrow \hat{\varphi}_m \xrightarrow{S} 0$.

Proof

Know that $\mathcal{F}: S(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. Lemma 3.2

$$|\lambda^\alpha D^\beta \hat{\varphi}(\lambda)| = \left| \int D^\alpha (x^\beta \varphi) e^{-i\lambda\cdot x}\,dx \right|$$
$$\leq \int |D^\alpha (x^\beta \varphi)|\,dx < \infty \quad (\dagger)$$

since $\varphi \in S(\mathbb{R}^n) \Rightarrow D^\alpha(x^\beta\varphi) \in S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

So $\|\hat{\varphi}\|_{\alpha,\beta} < \infty$ for all α, β i.e. $\hat{\varphi} \in S(\mathbb{R}^n)$

$\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$.

By suitably applying $(\dagger)$ to sequence $\varphi_m \xrightarrow{S} 0$, find $\varphi_m \xrightarrow{S} 0 \Rightarrow \hat{\varphi}_m \xrightarrow{S} 0$

Want to establish inversion. Consider

$$\int e^{i\lambda\cdot x} \hat{\varphi}(\lambda)\,d\lambda = \lim_{\varepsilon \downarrow 0} \int e^{i\lambda\cdot x} e^{-\varepsilon|\lambda|^2} \hat{\varphi}(\lambda)\,d\lambda \text{ (DCT)}$$

For all $\varepsilon > 0$, by Fubini:

$$= \int \varphi(y) \Big[\underbrace{\int e^{i\lambda(x-y) - \varepsilon|\lambda|^2} d\lambda}_{\prod_{j=1}^n \int e^{i\lambda_j(x_j - y_j) - \varepsilon\lambda_j^2} d\lambda_j} \Big] dy \quad (*)$$

$$= \int \varphi(y) \left[\prod_{j=1}^n \left(\frac{\pi}{\varepsilon}\right)^{1/2} e^{-\frac{(x_j-y_j)^2}{4\varepsilon}} \right] dy$$
$$= \int \varphi(y) \left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-|x-y|^2/4\varepsilon}\,dy.$$

$y' = \frac{x-y}{2\sqrt{\varepsilon}}$

$$= \int \varphi(x - 2\sqrt{\varepsilon}y) \pi^{n/2} \cdot 2^n e^{-|y|^2}\,dy$$

$$\xrightarrow[\text{(DCT)}]{\varepsilon \to 0} \varphi(x)(2\pi)^n \underbrace{\left(\sqrt{\tfrac{1}{\pi}}\right)^n \int e^{-|y|^2}\,dy}_{=1}$$

$$= (2\pi)^n \varphi(x).$$

i.e. $$\varphi(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} \hat{\varphi}(\lambda)\,d\lambda$$

$$\Leftrightarrow \varphi(-x) = \mathcal{F}\left[\frac{\hat{\varphi}}{(2\pi)^n}\right]$$

So get a bijection $\mathcal{F}: S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ and by previous, $\hat{\varphi}_m \xrightarrow{S} 0 \Rightarrow \varphi_m \xrightarrow{S} 0$. $\square$

$(*)$ $$\int_{\mathbb{R}} e^{i\lambda\sigma} e^{-\varepsilon\lambda^2} d\lambda = \int_{\mathbb{R}} e^{-\varepsilon(\lambda - \frac{i\sigma}{2\varepsilon})^2 - \frac{\sigma^2}{4\varepsilon}} d\lambda$$

$$= e^{-\sigma^2/4\varepsilon} \cdot \underbrace{\int_{\mathbb{R}} e^{-\varepsilon(\lambda - i\sigma/2\varepsilon)^2} d\lambda}$$

$\frac{\partial}{\partial\sigma}[\ \ ] = 0 \Rightarrow \sigma = 0$ wlog.

OR [Cauchy] $$\int_{\mathbb{R} + \frac{i\sigma}{2\varepsilon}} e^{-\varepsilon(z - i\sigma/2\varepsilon)^2} dz$$

$$= \int_{\mathbb{R}} e^{-\varepsilon x^2} dx = \sqrt{\frac{\pi}{\varepsilon}}$$

LECTURE 7

§ 3.3: Fourier Transform on $\mathcal{S}'(\mathbb{R}^n)$

To define F.T. on $\mathcal{S}'(\mathbb{R}^n)$ need Parseval.

Lemma 3.3:
If $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ then
$$\int \varphi(x)\hat{\psi}(x)\,dx = \int \hat{\varphi}(x)\psi(x)\,dx$$

Proof: by Fubini
$$LHS = \int \varphi(x)\left[\int e^{-i\lambda\cdot x}\psi(\lambda)\,d\lambda\right]dx$$
$$= \int \psi(\lambda)\left[\int e^{-i\lambda\cdot x}\varphi(x)\,dx\right]d\lambda$$
$$= \int \psi(\lambda)\hat{\varphi}(\lambda)\,d\lambda \qquad \square$$

If $u \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, then previous lemma states
$$\langle \hat{u}, \varphi\rangle = \langle u, \hat{\varphi}\rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$
Since $\mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, the RHS is well-defined for any $u \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 3.4:
For $u \in \mathcal{S}'(\mathbb{R}^n)$ define $\hat{u}$ by
$$\langle \hat{u}, \varphi\rangle := \langle u, \hat{\varphi}\rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

Take $u = \delta_0$.
$$\langle \hat{\delta}_0, \varphi\rangle = \langle \delta_0, \hat{\varphi}\rangle = \hat{\varphi}(0) = \int \varphi(x)\,dx$$
$$= \langle 1, \varphi\rangle$$
i.e. $\hat{\delta}_0 = 1$ in $\mathcal{S}'(\mathbb{R}^n)$. If $u = 1$
$\hookrightarrow \langle u, \varphi\rangle = \int u\varphi$
$$\langle \hat{1}, \varphi\rangle = \langle 1, \hat{\varphi}\rangle = \int \hat{\varphi}\,d\lambda = (2\pi)^n\varphi(0).$$
$$= \langle (2\pi)^n\delta_0, \varphi\rangle$$
In "old" language,
$$\text{"}\ \delta_0(x) = \frac{1}{(2\pi)^n}\int e^{-i\lambda\cdot x}\,d\lambda\ \text{"}$$

Straightforward to extend lemma 3.2 to $\mathcal{S}'(\mathbb{R}^n)$, i.e.
$$(D^\alpha u)^\wedge = \lambda^\alpha \hat{u} \qquad \text{(Check!)}$$
$$(x^\beta u)^\wedge = (-D)^\beta \hat{u}$$

Theorem 3.2:
The Fourier Transform defines a continuous bijection $\mathcal{F}: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Proof:
To get bijection, note
$$\check{u} = \frac{1}{(2\pi)^n}(\hat{u})^\wedge \qquad (*)$$
since $\langle \check{u}, \varphi\rangle = \langle u, \check{\varphi}\rangle \overset{(*)}{=} \langle u, (2\pi)^{-n}(\hat{\varphi})^\wedge\rangle$
$$= \langle (2\pi)^{-n}(\hat{u})^\wedge, \varphi\rangle.$$
$$(*)\quad \varphi(-x) = \frac{1}{(2\pi)^n}\int e^{-i\lambda x}\hat{\varphi}(\lambda)\,d\lambda = (2\pi)^{-n}(\hat{\varphi})^\wedge$$
To see that $\mathcal{F}: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, note that $\varphi_m \to 0$ in $\mathcal{S}(\mathbb{R}^n) \Leftrightarrow \hat{\varphi}_m \to 0$ in $\mathcal{S}(\mathbb{R}^n)$, so $\langle \hat{u}, \varphi_m\rangle = \langle u, \hat{\varphi}_m\rangle \to 0$ as $m \to \infty$, so $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

For continuity, suppose $u_m \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$, i.e.
$$\langle u_m, \varphi\rangle \to 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$
$$\Leftrightarrow \langle u_m, \hat{\varphi}\rangle \to 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$
$$\Leftrightarrow \langle \hat{u}_m, \varphi\rangle \to 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$
i.e. $u_m \xrightarrow{\mathcal{S}'} 0 \Leftrightarrow \hat{u}_m \xrightarrow{\mathcal{S}'} 0 \qquad \square$

§ 3.4: Sobolev Space

Definition 3.5:
For $s \in \mathbb{R}$ define Sobolev Space $H^s(\mathbb{R}^n)$ to be the $u \in \mathcal{S}'(\mathbb{R}^n)$ for which $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ can be identified with a (measurable) $\lambda \longmapsto \hat{u}(\lambda)$ that satisfies
$$\|u\|_{H^s}^2 = \int (1+|\lambda|^2)^s|\hat{u}(\lambda)|^2\,d\lambda < \infty.$$

We will use notation
$$\langle\lambda\rangle = (1+|\lambda|^2)^{1/2}$$
So $\langle\lambda\rangle \sim |\lambda|$ as $|\lambda| \to \infty$. See that $u \in H^s(\mathbb{R}^n)$ then $\langle\lambda\rangle^s\hat{u} \in L^2(\mathbb{R}^n)$.

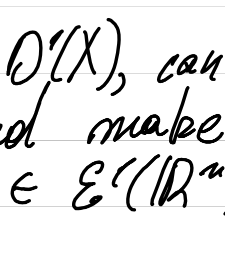
Lemma 3.4:
If $u \in H^s(\mathbb{R}^n)$ and $s > n/2$, then $u \in C(\mathbb{R}^n)$.

Proof:
We establish that $\hat{u} \in L^1(\mathbb{R}^n)$
$$\int |\hat{u}(\lambda)|\,d\lambda = \int \langle\lambda\rangle^{-s}\langle\lambda\rangle^{s}|\hat{u}|\,d\lambda$$
$$\leq \left[\int \langle\lambda\rangle^{-2s}\,d\lambda\right]^{1/2}\cdot\left[\int \langle\lambda\rangle^{2s}|\hat{u}(\lambda)|\,d\lambda\right]^{1/2}$$
$$= \left[\int_{S^{n-1}} d\sigma \underbrace{\int_0^\infty (1+r^2)^{-s}r^{n-1}\,dr}_{(+)}\right]^{1/2}\cdot \|u\|_{H^s(\mathbb{R}^n)}.$$
Note $(+) = O(r^{-2s+n-1})$ as $r \to \infty$
So integral is finite if $s > n/2$. CANNOT invoke inverse F.T. - only proved that inversion works on $\mathcal{S}(\mathbb{R}^n)$
$$\langle u, \hat{\varphi}\rangle = \langle \hat{u}, \varphi\rangle = \int \hat{u}(\lambda)\varphi(\lambda)\,d\lambda.$$
$$= \int \hat{u}(\lambda)\cdot\left[\frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\hat{\varphi}(x)\,dx\right]d\lambda$$
Since $\hat{u} \in L^1(\mathbb{R}^n)$, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can apply Fubini:
$$\langle u, \hat{\varphi}\rangle = \int \hat{\varphi}(x)\left[\frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\hat{u}(\lambda)\,d\lambda\right]dx$$
$$= \int u(x)\hat{\varphi}(x)\,dx$$
where $u(x) = \frac{1}{(2\pi)^n}\int e^{i\lambda x}\hat{u}(\lambda)\,d\lambda.$
Since $\hat{u} \in L^1(\mathbb{R}^n)$, by Dominated Convergence thm $\Rightarrow u \in C(\mathbb{R}^n) \qquad \square$

Corollary 3.1:
If $u \in H^s(\mathbb{R}^n)$ for ALL $s > n/2$ then $u \in C^\infty(\mathbb{R}^n)$.

[Replace u with $D^\alpha u$, show $\underset{\lambda^\alpha\hat{u}}{\underset{\parallel}{(D^\alpha u)^\wedge}} \in L^1(\mathbb{R}^n)$ etc, conclude $D^\alpha u \in C(\mathbb{R}^n)$]

[When understanding regularity, which is a LOCAL concept, can confine attention to φu, $\varphi \in \mathcal{D}(\mathbb{R}^n)$.



$\operatorname{supp}\varphi \subset \{|x - x_0| < \varepsilon\}$.

So very rarely need to study u in isolation, φu, $\varphi \in \mathcal{D}(\mathbb{R}^n)$, will do.

So if $u \in \mathcal{D}'(X)$, can consider $\varphi u \in \mathcal{D}'(X)$, $\varphi \in \mathcal{D}(X)$, and make extension
$$(\varphi u)_{ext} \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

See $H^s_{loc}(X)$ in next lecture]

LECTURE 8

Def^n 3.6:
Say $u \in \mathcal{D}'(X)$ belongs to the <u>local</u> Sobolev Space $H^s_{loc}(X)$ if $u\varphi$ (extends to) an <u>element</u> of $H^s(\mathbb{R}^n)$ for each $\varphi \in \mathcal{D}(X)$.

Note we interpret $\varphi u \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ by $\langle \varphi u, \psi \rangle := \langle u, \varphi\psi \rangle \quad \forall \psi \in \mathcal{E}(\mathbb{R}^n)$.
Well-defined since $\mathrm{supp}(\varphi\psi) \subset X$.

§ 4: Applications of Fourier Transform

§ 4.1: Elliptic Regularity

Interested in problems of form
$$P(D)u = f$$
where $u, f \in \mathcal{D}'(X)$, where P is a polynomial in λ, e.g. $p(\lambda) = \lambda_1^2 + \dots + \lambda_n^2$
$$P(D) = -\left(\frac{\partial}{\partial x_1}\right)^2 - \dots - \left(\frac{\partial}{\partial x_n}\right)^2 = -\Delta.$$

Interested in: if $f \in H^s_{loc}(X)$, can we say that $u \in H^t_{loc}(X)$ for some $t = t(s, P)$?
Answer this when $P(D)$ is elliptic.

Definition 4.1:
An N^{th} order P.D.O. constant coeff's.
$$P(D) = \sum_{|\alpha| \le N} c_\alpha D^\alpha$$
has <u>principal symbol</u> defined by
$$\sigma_p(\lambda) = \sum_{|\alpha| = N} c_\alpha \lambda^\alpha.$$

Say $P(D)$ is <u>elliptic</u> if $\sigma_p(\lambda) \neq 0$ on $\mathbb{R}^n \setminus \{0\}$.

[$\hat{u} = \frac{\hat{f}}{p(\lambda)}$?, $\quad p(\lambda) \simeq \sigma_p(\lambda), |\lambda| \to \infty$]

Lemma 4.1:
If $P(D)$ is N^{th} order elliptic then for $|\lambda|$ sufficiently large $|P(\lambda)| \gtrsim \langle \lambda \rangle^N$

<u>Proof:</u> By continuity & compactness, since $\sigma_p(\lambda) \neq 0$ on S^{n-1}:
$$\min_{|\lambda|=1} |\sigma_p(\lambda)| = C > 0$$
Then for $\lambda \in \mathbb{R}^n \setminus \{0\}$
$$|\sigma_p(\lambda)| = |\lambda|^N \left| \sum_{|\alpha|=N} c_\alpha \left(\frac{\lambda}{|\lambda|}\right)^\alpha \right|$$
$$\ge C|\lambda|^N$$
By triangle inequality:
$$|P(\lambda)| \ge |\sigma_p(\lambda)| - |P(\lambda) - \sigma_p(\lambda)|$$
$$\ge \left[C - \frac{|P(\lambda) - \sigma_p(\lambda)|}{|\lambda|^N} \right] |\lambda|^N$$
Since $|P(\lambda) - \sigma_p(\lambda)| = O(|\lambda|^{N-1})$. Choosing $|\lambda|$ sufficiently large so
$$\frac{|P(\lambda) - \sigma_p(\lambda)|}{|\lambda|^N} \le \frac{C}{2}$$
Hence, for $|\lambda|$ sufficiently large
$$|P(\lambda)| \ge \frac{C}{2} |\lambda|^N \gtrsim \langle \lambda \rangle^N$$ □

Theorem 4.1:

If $P(D)u$ is N^{th} order elliptic and $P(D)u \in H^s_{loc}(X) \implies u \in H^{s+N}_{loc}(X)$.

Today prove easier version, relevant if $u \in \mathcal{E}'(\mathbb{R}^n)$. Use fact [if $u \in \mathcal{E}'(\mathbb{R}^n)$, then $\hat{u} \in \mathcal{E}(\mathbb{R}^n)$, $|\hat{u}(\lambda)| \lesssim \langle \lambda \rangle^M$ some $M \ge 0$]

When $u \in \mathcal{E}'(\mathbb{R}^n)$, can use <u>parametrix</u> to prove version of Th^m 4.1.

Def^n 4.2:
Say that $E \in \mathcal{D}'(\mathbb{R}^n)$ is a parametrix for $P(D)$ if exists $w \in \mathcal{E}(\mathbb{R}^n)$ s.t.
$$P(D)E = \delta_0 + w$$

[$P(D)G = \delta_0 \qquad Lu = f \Rightarrow y = G * f$]

Lemma 4.2:
Every (non-zero), <u>elliptic</u> $P(D)$ admits a parametrix $E \in \mathcal{E}(\mathbb{R}^n \setminus \{0\})$.
<u>Proof:</u>
Fix $R > 0$ s.t. $|P(\lambda)| \gtrsim \langle \lambda \rangle^N$ on $|\lambda| > R$, fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ such that $\chi = 1$ on $|\lambda| \le R$ and $\chi = 0$ on $|\lambda| > R+1$.

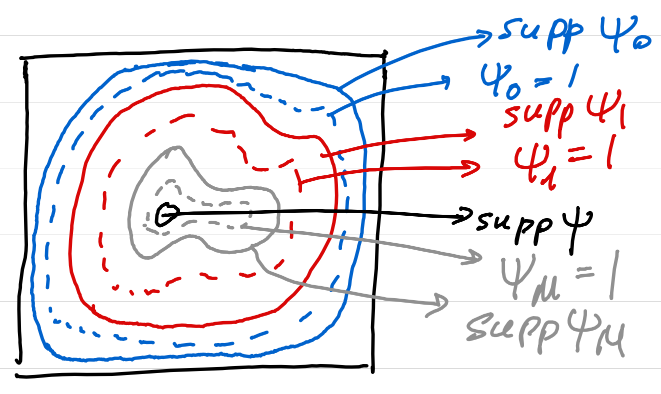
Define $E \in \mathcal{S}'(\mathbb{R}^n)$ by
$$\hat{E}(\lambda) = \frac{(1 - \chi(\lambda))}{P(\lambda)}$$
Then $\hat{E}$ is smooth and $|\hat{E}| \lesssim \langle \lambda \rangle^{-N}$ for $|\lambda|$, so $\hat{E} \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow E \in \mathcal{S}'(\mathbb{R}^n)$. By Inverse F.T.
$$P(D)E = \delta_0 + w$$
where $\hat{w} = -\chi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow w \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$.
For $|\lambda| > R+1$ have:
$$|(x^\beta E)^\wedge(\lambda)| = |D^\beta \hat{E}(\lambda)|$$
$$= \left| D^\beta \left(\frac{1}{P(\lambda)} \right) \right|$$
$$\lesssim \langle \lambda \rangle^{-N-|\beta|} \quad \text{[induction]}$$
So for <u>every</u> $s \in \mathbb{R}$ [particularly $s > n/2$] there is a β multi-index s.t. $x^\beta E \in H^s(\mathbb{R}^n)$.
So for each α, $D^\alpha(x^\beta E)$ is cts for $|\beta|$ sufficiently large [Sobolev Lemma].
I.e. E is smooth away from $x = 0$, $E \in \mathcal{E}(\mathbb{R}^n \setminus \{0\})$. (*) □

Proof of easy Th^m 4.1: if $u \in \mathcal{E}'(\mathbb{R}^n)$, then $\hat{u} \in \mathcal{E}(\mathbb{R}^n)$, using
$$P(\lambda)\hat{E}(\lambda) = 1 + \hat{w}$$
i.e. $\quad 1 = P(\lambda)\hat{E} - \hat{w} \qquad \varphi \in \mathcal{D}(\mathbb{R}^n) \Big\} = o(\langle \lambda \rangle^{-k}) \,\forall k$
$$\Rightarrow \quad \hat{u} = [P(\lambda)\hat{u}]\hat{E} - \hat{w}\hat{u}$$
$P(D)u \in H^s(\mathbb{R}^n)$ and $P(\lambda)\hat{u}\langle \lambda \rangle^s \in L^2(\mathbb{R}^n)$.
$$\Rightarrow \quad \langle \lambda \rangle^{s+N}\hat{u} = [P(\lambda)\hat{u}\langle \lambda \rangle^s] \overbrace{\hat{E}(\lambda)\langle \lambda \rangle^N}^{\lesssim 1} - \underbrace{\hat{w}\hat{u}\langle \lambda \rangle^{s+N}}_{o(\langle \lambda \rangle^{-k})\,\forall k}$$
So $\|\hat{u}\langle \lambda \rangle^{s+N}\|_{L^2} \lesssim 1$
$$\Longrightarrow \quad u \in H^{s+N}(\mathbb{R}^n)$$

[$P(D)u = f \quad f \in H^s_{loc}(X), \; \varphi f \in H^s(\mathbb{R}^n) \; \varphi \in \mathcal{D}(X)$.
$LHS = P(D)[\varphi u] + \underbrace{[\varphi, P(D)]}_{\text{ord } N-1}(u)$]

(*) i.e. $E|_{\mathbb{R}^n \setminus \{0\}} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$
$$\cap$$
$$C^\infty(\mathbb{R}^n \setminus \{0\}).$$
Since $\forall s > 0, \exists \beta \in \mathbb{N}^n$ s.t. $x^\beta E \in H^s(\mathbb{R}^n)$
$\Rightarrow E = \frac{x^\beta E}{x^\beta}$ is smooth since $\bigcap_{s>0} H^s(\mathbb{R}^n) \supset C^\infty(\mathbb{R}^n)$.

<u>LECTURE 9</u>



<u>Proof</u> (Thm 4.1):
Use following (ES2):
- $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\exists t \in \mathbb{R}$ s.t. $u \in H^t(\mathbb{R}^n)$.
- If $u \in H^s(\mathbb{R}^n) \Rightarrow D^\alpha u \in H^{s-|\alpha|}(\mathbb{R}^n)$.
- If $s \geq t \Rightarrow H^s(\mathbb{R}^n) \subset H^t(\mathbb{R}^n)$.
- If $\varphi \in S(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n)$ $\Rightarrow$ $\varphi u \in H^s(\mathbb{R}^n)$.

Fix $\varphi \in D(X)$. Introduce ~~tool~~ functions $\{\psi_0, \psi_1, \dots, \psi_M\}$ $[\psi_i \in D(X)]$ s.t. $\psi_{i-1} = 1$ on supp ψ_i and supp $\varphi \subset$ supp $\psi_M \subset \dots \subset$ supp ψ_0.
Note that $\psi_0 u \in \mathcal{E}'(\mathbb{R}^n)$, so $\psi_0 u \in H^t(\mathbb{R}^n)$ (some t).
Then $P(D)[\psi_1 u] = \psi_1 P(D)u + [P(D), \psi_1](u)$
$= \psi_1 P(D)u + [P(D), \psi_1](\psi_0 u)$ since $\psi_0 = 1$ on supp ψ_1.
$\in H^s(\mathbb{R}^n) \oplus H^{t-N+1}(\mathbb{R}^n)$.
i.e. $P(D)[\psi_1 u] \in H^{\tilde{A}_1}(\mathbb{R}^n)$, where
$\tilde{A}_1 = \min\{s, t-N+1\}$ i.e.

$$(\dagger) \int \langle\lambda\rangle^{2\tilde{A}_1} |P(\lambda)[\psi_1 u]^\wedge(\lambda)|^2 d\lambda < \infty$$

Since $|P(\lambda)| \gtrsim \langle\lambda\rangle^N$ for $|\lambda|$ sufficiently large.

$$(\dagger) \Rightarrow \int \langle\lambda\rangle^{2(\tilde{A}_1+N)} |[\psi_1 u]^\wedge(\lambda)|^2 d\lambda$$

$$\lesssim \int \langle\lambda\rangle^{2\tilde{A}_1} |P(\lambda)[\psi_1 u]^\wedge(\lambda)|^2 d\lambda < \infty.$$

i.e. $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$, $A_1 = \tilde{A}_1 + N = \min\{s+N, t+1\}$.
Similarly,

$$P(D)[\psi_2 u] = \psi_2 P(D)u + [P(D), \psi_2](u)$$
$$= \psi_2 P(D)u + [P(D), \psi_2](\psi_1 u)$$

since $\psi_1 = 1$ on supp ψ_2

$$\Rightarrow \psi_2 u \in H^{A_2}(\mathbb{R}^n),\ A_2 = \min\{s+N, \underbrace{\min\{s+N+1, t+2\}}_{A_1+1}\}$$
$$= \min\{s+N, t+2\}$$

Proceeding inductively,
$$\psi_M u \in H^{A_M}(\mathbb{R}^n),$$
$$A_M = \min\{s+N, t+M\}.$$
Choose $M > s+N-t$, so $A_M = s+N$.
Since $\psi_M = 1$ on supp φ get $\varphi u \in H^{s+N}(\mathbb{R}^n)$.
Since $\varphi \in D(X)$ arbitrary, conclude $u \in H^{s+N}_{loc}(X)$. $\square$

<u>§ 4.2: Fundamental Solutions</u>

To solve probs of form $P(D)u = f$, can use fundamental sol^n:

<u>Def^n 4.3:</u>
Say $E \in D'(\mathbb{R}^n)$ is fundamental sol^n for $P(D)$ if $P(D)E = \delta$.

<u>Lemma 4.3:</u>
Fundamental sol^n for

$$P(D) = \frac{\partial}{\partial \bar{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$$ is

given by $E = \frac{1}{\pi z}$. $(z = x_1 + i x_2 \in \mathbb{C} \simeq \mathbb{R}^2)$.

<u>Proof:</u>
$E \in L^1_{loc}(\mathbb{R}^2)$, so $E \in D'(\mathbb{R}^2)$. For $\varphi \in D(\mathbb{R}^2)$

$$\left\langle \frac{\partial}{\partial \bar{z}} E, \varphi \right\rangle = -\left\langle E, \frac{\partial \varphi}{\partial \bar{z}} \right\rangle$$

$$= -\lim_{\varepsilon \downarrow 0} \int_{|z|>\varepsilon} \frac{1}{\pi z} \frac{\partial \varphi}{\partial \bar{z}} dx \quad \text{(Dominated Convergence)}$$

$$= -\lim_{\varepsilon \to 0} \int_{|z|>\varepsilon} \frac{\partial}{\partial \bar{z}}\left(\frac{\varphi}{\pi z}\right) dx \quad \text{Green's Theorem}$$

$$\left[\int_D \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2}\right) dx_1 dx_2 = \int_{\partial D} P\,dx_1 + Q\,dx_2 \right]$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{\varphi}{z} dz \quad (dz = dx_1 + i\,dx_2).$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\varphi(\varepsilon\cos\theta, \varepsilon\sin\theta)\, i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} d\theta$$

$$= \frac{1}{2\pi} \cdot 2\pi\, \varphi(0) = \langle \delta_0, \varphi \rangle \quad \text{(by DC again)}.$$

<u>Lemma 4.4:</u>
The fundamental Solution for the Heat operator

$$P(D) = \frac{\partial}{\partial t} - \Delta_x \quad \text{on } \mathbb{R} \times \mathbb{R}^n$$

is

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2} \exp\left[\frac{-|x|^2}{2t}\right], & t > 0 \\ 0, & t \leq 0 \end{cases}$$

<u>Proof:</u>
Note that $P(D)E = 0$ on $t \geq \varepsilon > 0$.
For $\varphi \in D(\mathbb{R}^{n+1})$:

$$\langle (\partial/\partial t - \Delta_x)E, \varphi \rangle = -\left\langle E, \left(\frac{\partial}{\partial t} + \Delta_x\right)\varphi \right\rangle$$

$$= \lim_{\varepsilon \to 0} -\int_\varepsilon^\infty dt \int_{\mathbb{R}^n} dx\, E(x,t)[\varphi_t + \Delta_x \varphi]$$

$$= \lim_{\varepsilon \to 0} -\int_{\mathbb{R}^n} dx\, E\varphi \Big|_{t=\varepsilon}^\infty + \int_\varepsilon^\infty dt \int_{\mathbb{R}^n} dx\, \varphi\,\underbrace{[E_t - \Delta_x E]}_{=0}$$

$$= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} dx\, (4\pi\varepsilon)^{-n/2} e^{-|x|^2/2\varepsilon} \varphi(x, \varepsilon), \quad \frac{x}{\sqrt{2\varepsilon}} = y$$

$$= \lim_{\varepsilon \to 0} \int dy\, (2\varepsilon)^{n/2} (4\pi\varepsilon)^{-n/2} e^{-|y|^2} \varphi(\sqrt{2\varepsilon}\, y, \varepsilon)$$

$$(DC) = \varphi(0) \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2} dy = \varphi(0) = \langle \delta_0, \varphi \rangle \quad \square$$

$$\left[\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \int \frac{\hat{\varphi}(-\lambda)}{P(\lambda)} d\lambda \right.$$

$$\langle P(D)E, \varphi \rangle = \langle E, P(-D)\varphi \rangle$$

$$\left. = \frac{1}{(2\pi)^n} \int \frac{P(\lambda)\hat{\varphi}(-\lambda)}{P(\lambda)} d\lambda = \varphi(0). \right]$$

Hörmander's Staircase $\rightarrow$ Construction of fundamental sol^n

Try to construct surface
$\Sigma \subset \mathbb{C}^n$ s.t. $\Sigma \simeq \mathbb{R}^n$
(homotopic) and for which
$\langle E, \varphi \rangle = \frac{1}{(2\pi)^n} \int_\Sigma \frac{\hat{\varphi}(-\lambda)}{P(\lambda)} d\lambda, \ \varphi \in \mathcal{D}(\mathbb{R}^n)$
defines an element of $\mathcal{D}'(\mathbb{R}^n)$. Note then
$\langle P(D)E, \varphi \rangle = \langle E, P(-D)\varphi \rangle$

$$= \frac{1}{(2\pi)^n} \int_\Sigma \frac{P(\lambda)\hat{\varphi}(-\lambda)}{P(\lambda)} d\lambda$$
$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{\varphi}(-\lambda) d\lambda$$

Use complex analysis and $\Sigma \simeq \mathbb{R}^n$ $\hookrightarrow = \varphi(0)$

Call Σ "Hörmander's staircase."

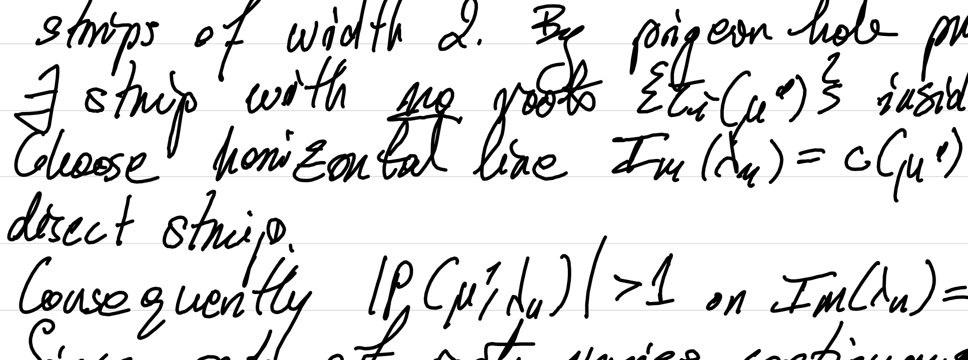
Lemma 4.5:
For $\lambda \in \mathbb{R}^n$ write $\lambda = (\lambda', \lambda^n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ etc.
For each $\lambda' \in \mathbb{R}^{n-1}$, if $\varphi \in \mathcal{D}(\mathbb{R}^n)$ then
$\mathbb{C} \ni z \longmapsto \hat{\varphi}(\lambda', z)$ is holomorphic and
$\exists\, \delta > 0$ such that
$|\hat{\varphi}(\lambda', z)| \lesssim_m (1+|z|)^{-m} e^{\delta |\operatorname{Im} z|}, \ m = 0, 1, 2 \dots$

I.e. have fast decay at horizontal infinity, so
$\int_{\mathbb{R}+i\eta} \hat{\varphi}(\lambda', z)\, dz = \int_{\mathbb{R}} \hat{\varphi}(\lambda', \lambda_n)\, d\lambda_n \quad \forall \eta \in \mathbb{R}.$
(Cauchy's Theorem)

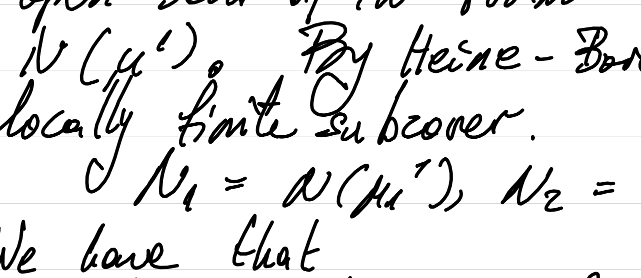
Theorem 4.2:
For every non-zero $P(D)$ there exists a fundamental solution.

Proof:
By scaling and rotating (fully) coord axes, can assume $P(\lambda)$ has the form:
$P(\lambda', \lambda_n) = \lambda_n^M + \sum_{m=0}^{M-1} a_m(\lambda')\lambda_n^m$
Let us FIX $\mu' \in \mathbb{R}^{n-1}$. Then
$P(\mu'; \lambda_n) = \prod_{i=1}^{M} (\lambda_n - \tau_i(\mu'))$ where $\{\tau_i(\mu')\}$
are the zeros of the polynomial $\lambda_n \mapsto P(\mu', \lambda_n)$.

Claim: there exists a horizontal line
$\operatorname{Im} \lambda_n = c(\mu')$ in the complex λ_n-plane
inside the strip $|\operatorname{Im} \lambda_n| \le M+1$ such that
$|\operatorname{Im}(\lambda_n - \tau_i(\mu'))| > 1, \ i = 1, 2, \dots, M.$



Indeed, $|\operatorname{Im}(\lambda_n)| \le M+1$ consists of $M+1$ strips of width 2. By pigeon hole principle, $\exists$ strip with no roots $\{\tau_i(\mu')\}$ inside it. Choose horizontal line $\operatorname{Im}(\lambda_n) = c(\mu')$ to direct strip.
Consequently $|P(\mu'; \lambda_n)| > 1$ on $\operatorname{Im}(\lambda_n) = c(\mu')$.
Since set of roots varies continuously with the coefficients of polynomial, deduce that same statement holds for λ' in a sufficiently small nbhd of μ', call it $N(\mu')$. So get: open
$|P(\lambda', \lambda_n)| > 1$ for $\begin{cases} \operatorname{Im}(\lambda_n) = c(\mu') \\ \lambda' \in N(\mu'). \end{cases}$



Can do this for every $\mu' \in \mathbb{R}^{n-1}$, can generate an open cover of $\mathbb{R}^{n-1}$ with open sets of the form
$N(\mu')$. By Heine-Borel can extract locally finite subcover.
$N_1 = N(\mu_1'), \ N_2 = N(\mu_2'), \dots$
We have that
$|P(\lambda', \lambda_n)| > 1$ on $\begin{cases} \operatorname{Im}(\lambda_n) = c_i = c(\mu_i') \\ \lambda' \in N_i. \end{cases}$

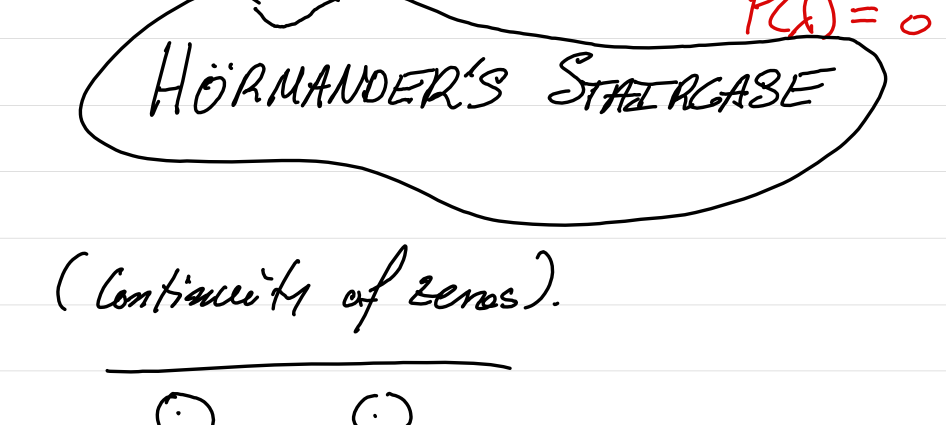
Define open sets
$\Delta_1 = N_1$
$\Delta_i := N_i \setminus (\overline{N}_1 \cup \dots \cup \overline{N}_{i-1}).$
Have $\{\Delta_i\}$ are open, disjoint,
$\bigcup_{i=1}^{\infty} \overline{\Delta}_i = \mathbb{R}^{n-1}$ and $|P(\lambda', \lambda_n)| > 1$ on $\begin{cases} \operatorname{Im} \lambda_n = c_i \\ \lambda' \in \Delta_i. \end{cases}$

Define, for $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

$$\langle E, \varphi \rangle := \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \left[\int_{\operatorname{Im} \lambda_n = c_i} \frac{\hat{\varphi}(-\lambda', -\lambda_n)\, d\lambda_n}{P(\lambda', \lambda_n)} \right]$$

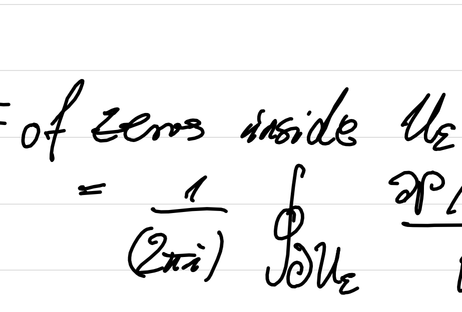
Then
$$\langle P(D)E, \varphi \rangle = \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\operatorname{Im} \lambda_n = c_i} \hat{\varphi}(-\lambda', -\lambda_n)\, d\lambda_n$$
$$= (\text{By Cauchy + Lemma 4.5})$$
$$= \frac{1}{(2\pi)^n} \sum_{i=1}^{\infty} \int_{\Delta_i} d\lambda' \int_{\mathbb{R}} d\lambda_n\, \hat{\varphi}(-\lambda', -\lambda_n)$$
$$(\text{no } i \text{ dependence}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{n-1}} d\lambda' \int_{\mathbb{R}} \hat{\varphi}(-\lambda', -\lambda_n)$$
$$= \varphi(0) = \langle \delta_0, \varphi \rangle$$

Still need to show that $E \in \mathcal{D}'(\mathbb{R}^n)$, see ES3.
So $P(D)E = \delta_0, \ E \in \mathcal{D}'(\mathbb{R}^n)$. $\square$



HÖRMANDER'S STAIRCASE

(Continuity of zeros).



$\mathcal{U}_\varepsilon$ nbhds, balls of radius ε. Centred around zeros of poly $\lambda_n \mapsto P(\mu'; \lambda_n)$. μ' fixed

\# of zeros inside $\mathcal{U}_\varepsilon$ (arg-principle)
$$= \frac{1}{(2\pi i)} \oint_{\partial \mathcal{U}_\varepsilon} \frac{\partial P / \partial \lambda_n (\mu', \lambda_n)\, d\lambda_n}{P(\mu', \lambda_n)}$$

is continuous in a nbhd of μ' and integer valued, hence constant.

LECTURE 11

§4.3: Structure Theorem for $\mathcal{E}'(X)$

We know that if $f \in C(X)$, then $\partial^\alpha f \in \mathcal{D}'(X)$ with $\langle \partial^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \int_X f \partial^\alpha \varphi \, dx \quad \forall \varphi \in \mathcal{D}(X)$.
Also, note that $\delta_0 = (xH)''$ in $\mathcal{D}'(\mathbb{R})$.
Natural to ask: can all distributions be written in the form
$$u = \sum_\alpha \partial^\alpha f_\alpha \quad \text{in } \mathcal{D}'(X) \text{ where}$$
$f_\alpha \in C(X)$? We will prove in case $\mathcal{E}'(X)$, but result is true more generally.

Lemma 4.6:

If $u \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, then $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ can be identified with the smooth (analytic) function $\lambda \mapsto \hat{u}(\lambda) = \langle u(x), e^{-i\lambda \cdot x} \rangle$.
Also, $\exists M \geq 0$ such that $|\hat{u}(\lambda)| \lesssim \langle \lambda \rangle^M$!

Proof:
Fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ with $\chi = \begin{cases} 1, & |x|<1 \\ 0, & |x|>2 \end{cases}$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set $\varphi_m = \chi(x/m)\varphi(x) \in \mathcal{D}(X)$. Claim:
$\varphi_m \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. For arbitrary α, β:
$$\|\varphi - \varphi_m\|_{\alpha,\beta} = \|x^\alpha D^\beta [\varphi(x) \cdot (1 - \chi(x/m))]\|_\infty.$$
$$= \Big\|x^\alpha \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} D^\gamma \varphi \, D^{\beta-\gamma}(1 - \chi(x/m))\Big\|_\infty$$
All derivatives of $x \mapsto (1-\chi(x/m))$ tend to zero uniformly and
$$\|x^\alpha D^\beta \varphi \cdot (1 - \chi(x/m))\|_\infty \qquad 1 < \frac{|x|}{m}$$
$$\lesssim \sup_{|x| \geq m} \Big|x^\alpha D^\beta \varphi \cdot \frac{|x|}{m}\Big|$$
$$\lesssim \frac{\|\varphi\|_{\alpha+1,\beta}}{m} \longrightarrow 0$$

So by sequential continuity of $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.
$$\langle \hat{u}, \varphi \rangle = \lim_{m\to\infty} \langle \hat{u}, \varphi_m \rangle = \lim_{m\to\infty} \langle u, \hat{\varphi}_m \rangle$$
$$\stackrel{\circ}{=} \lim_{m\to\infty} \Big\langle u(x), \int e^{-i\lambda\cdot x} \varphi_m(\lambda) \, d\lambda \Big\rangle$$
By Riemann sum argument in Lemma 1.5:
[note each φ_m has compact support],
$$\stackrel{\circ}{=} \lim_{m\to\infty} \int \langle u(x), e^{-i\lambda\cdot x} \rangle \varphi_m(\lambda) \, d\lambda$$
Since power series for $x \mapsto e^{-i\lambda \cdot x}$ converges locally uniformly, can interchange $\langle \cdot, \cdot \rangle$ with infinite sum, by sequential continuity.
So $\hat{u}(\lambda) = \langle u(x), e^{-i\lambda\cdot x} \rangle$ is smooth and by semi-norm estimate of $u \in \mathcal{E}'(\mathbb{R}^n)$, $\exists$ $C, N \geq 0$ and compact $K \subset \mathbb{R}^n$ s.t.
$$|\hat{u}(\lambda)| = |\langle u(x), e^{-i\lambda\cdot x} \rangle|$$
$$\leq C \sum_{|\alpha| \leq N} \sup_K |\partial_x^\alpha (e^{-i\lambda \cdot x})|$$
$$\lesssim \langle \lambda \rangle^N, \quad \lambda \in \mathbb{R}^n.$$
So by DCT $\langle \hat{u}, \varphi \rangle = \int \hat{u}(\lambda) \varphi(\lambda) \, d\lambda$, i.e. $\hat{u}$ can be identified with $\lambda \mapsto \hat{u}(\lambda)$ $\square$

Theorem 4.3:

For each $u \in \mathcal{E}'(X)$, there exists a finite collection $\{f_\alpha\}$, $f_\alpha \in C(X)$ and $\text{supp}(f_\alpha) \subset X$, such that $u = \sum_\alpha \partial^\alpha f_\alpha$ in $\mathcal{E}'(X)$.

Proof:
Fix $\rho \in \mathcal{D}(X)$ such that $\rho = 1$ on $\text{supp}(u)$. Then for $\varphi \in \mathcal{E}(X)$ have:
$$\langle u, \varphi \rangle = \langle u, \rho\varphi \rangle + \langle u, (1-\rho)\varphi \rangle$$
$$= \langle u, \rho\varphi \rangle$$
By extending by zero, can treat $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\rho\varphi \in \mathcal{D}(\mathbb{R}^n)$. For some $\psi \in \mathcal{S}(\mathbb{R}^n)$, can write $(\rho\varphi) = (\hat{\psi})^\wedge$. In fact
$$(2\pi)^n \check{\psi} = \rho\varphi \quad (*)$$
So we have
$$\langle u, \varphi \rangle = \langle u, (\hat{\psi})^\wedge \rangle = \langle \hat{u}, \hat{\psi} \rangle$$
Note that $([1-\Delta]^m \psi)^\wedge(\lambda) = \langle \lambda \rangle^{2m} \hat{\psi}(\lambda)$ where $\Delta \equiv \sum_i (\partial/\partial x_i)^2$.

$$\langle u, \varphi \rangle = \langle \langle \lambda \rangle^{-2m} \hat{u}, [(1-\Delta)^m \psi]^\wedge \rangle.$$
By choosing m sufficiently large and defining
$$f(x) = \frac{1}{(2\pi)^n} \int e^{i\lambda\cdot x} \underbrace{\langle \lambda \rangle^{-2m} \hat{u}(\lambda)}_{\in L^1(\mathbb{R}^n)} \, d\lambda.$$

we see, that by DCT, that $f \in C(\mathbb{R}^n)$. Also, $(2\pi)^n \check{f} = (\langle \lambda \rangle^{-2m} \hat{u})^\wedge$. Hence
$$\langle u, \varphi \rangle = \langle (\langle \lambda \rangle^{-2m} \hat{u})^\wedge, [I-\Delta]^m \psi \rangle$$
$$= \langle (2\pi)^n \check{f}, [I-\Delta]^m \psi \rangle$$
$$\stackrel{\circ}{=} \langle f, [1-\Delta]^m [(2\pi)^n \check{\psi}] \rangle$$
and from $(*)$ $\stackrel{\circ}{=} \langle f, [I-\Delta]^m (\rho\varphi) \rangle$
Can expand derivatives, by Leibniz:
$$\langle u, \varphi \rangle = \Big\langle f, \sum_\alpha (-1)^{|\alpha|} \rho_\alpha \partial^\alpha \varphi \Big\rangle \text{ where}$$

$\rho_\alpha \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(\rho_\alpha) \subset X$. So
$$\langle u, \varphi \rangle = \Big\langle \sum_\alpha \partial^\alpha (\rho_\alpha f), \varphi \Big\rangle$$
$$\equiv \Big\langle \sum_\alpha \partial^\alpha f_\alpha, \varphi \Big\rangle$$
where $f_\alpha \in C(X)$ and $\text{supp}(f_\alpha) \subset X$. $\square$

Example:
Know that $\delta_0 = (xH)''$ $\to$ $\varphi \equiv 1$ on some nbhd of 0.
- If $\varphi(0) = 1$, $\varphi \in \mathcal{D}(\mathbb{R})$ then $\delta_0 = \varphi\delta_0$
- $\langle \delta_0, f \rangle$, $f \in \mathcal{D}(\mathbb{R})$.

$\stackrel{\circ}{=} \langle \varphi (xH)'', f \rangle$
$$(\varphi(xH))'' = \varphi''(xH) + 2\varphi'(xH)' + \varphi(xH)''$$

$$\varphi'(xH)' = (\varphi' xH)' - \varphi'' xH$$
$$= -\varphi''(xH) + 2(\varphi' xH)' + \varphi(xH)''$$
$$\Rightarrow \quad \varphi(xH)'' = (\varphi xH)'' + \varphi''(xH) - 2(\varphi' xH)'$$

$$\stackrel{\circ}{=} \langle (\varphi xH)'' + \varphi''(xH) - 2(\varphi'(xH))', f \rangle.$$
$$\Rightarrow \delta_0 = (\underbrace{\varphi xH}_{\in C_c(\mathbb{R})})'' + \varphi\underbrace{''(xH)}_{\in C_c(\mathbb{R})} - 2(\underbrace{\varphi'(xH)'}_{C_c(\mathbb{R})})'$$
since $\varphi' \equiv 0$ on some nbhd of 0 and $(xH)' = xH' + H = H$ as a distribution.

NOTE:

- Existence of fundamental solution theorem (Hörmander Staircase) is called "Malgrange-Ehrenpreis Theorem."

§ 4.4: Paley-Wiener-Schwartz Theorem:

Have seen that if $u \in \mathcal{E}'(\mathbb{R}^n)$, then $\hat{u}$ can be identified with $\lambda \mapsto \hat{u}(\lambda) = \langle u(x), e^{-i\lambda\cdot x}\rangle$

Take complex analytic extension to $z \in \mathbb{C}^n$, call $\hat{u}(z) = \langle u(x), e^{-iz\cdot x}\rangle$ the Fourier-Laplace transform of $u \in \mathcal{E}'(\mathbb{R})$. Know $\exists C, N \geq 0$, $K \Subset \mathbb{R}^n$ compact: $|\hat{u}(z)| = |\langle u(x), e^{-iz\cdot x}\rangle|$

$$\leq C \cdot \sum_{|\alpha|\leq N} \sup_K |\partial_x^\alpha (e^{-iz\cdot x})|.$$

Also $z \mapsto \hat{u}(z)$ is entire [power series of $x \mapsto e^{-iz\cdot x}$ converges locally uniformly, so can apply u termwise (sequential continuity of u) to get power series for $\hat{u}(z)$].

Lemma 4.7:

If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\mathrm{supp}(u) \subset \overline{B_\delta} = \{x \in \mathbb{R}^n : |x| \leq \delta\}$ then $\exists C, N \geq 0$ such that

$$|\hat{u}(z)| \leq C \cdot (1+|z|)^N e^{\delta \cdot |\mathrm{Im}\, z|}, \quad z \in \mathbb{C}.$$

Proof:

Fix $\psi \in C^\infty(\mathbb{R})$ such that $\psi(\tau) = 1$ on $\tau \geq -1/2$, $\psi(\tau) = 0$, on $\tau \leq -1$.

For $\varepsilon > 0$, define $\varphi_\varepsilon(x) = \psi(\varepsilon(\delta - |x|))$, $x \in \mathbb{R}^n$

Then $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and

$$\begin{cases} \varphi_\varepsilon = 1 & \text{on } |x| \leq \delta + \frac{1}{2\varepsilon} \\ \varphi_\varepsilon = 0 & \text{on } |x| \geq \delta + 1/\varepsilon. \end{cases}$$

Note that $\varphi_\varepsilon = 1$ on $\mathrm{supp}(u)$. Since $u \in \mathcal{E}'(\mathbb{R}^n)$, $\exists C, N \geq 0$ s.t. $|\hat{u}(z)| = |\langle u(x), \varphi_\varepsilon(x) \cdot e^{-iz\cdot x}\rangle|$

$$\leq C \cdot \sum_{|\alpha|\leq N} \sup_{|x|\leq \delta+1/\varepsilon} |\partial_x^\alpha [\varphi_\varepsilon e^{-ix\cdot z}]|.$$

Note $|\partial^\beta \varphi_\varepsilon| \lesssim_\beta \varepsilon^{|\beta|}$ and $|\partial^\gamma e^{-iz\cdot x}| \lesssim |z|^{|\gamma|} e^{(\delta + 1/\varepsilon)|\mathrm{Im}\, z|}$ on $\mathrm{supp}\, \varphi_\varepsilon$.

$$\Longrightarrow |\hat{u}(z)| \lesssim \sum_{|\beta|+|\gamma|\leq N} \varepsilon^{|\beta|} |z|^{|\gamma|} e^{(\delta+1/\varepsilon)\cdot|\mathrm{Im}\, z|}$$

Take $\varepsilon = (|z|+1) \Rightarrow$ result $\square$

Paley-Wiener-Schwartz is about converse: if $z \mapsto U(z)$ is entire function of $z \in \mathbb{C}^n$ and $|U(z)| \lesssim (1+|z|)^N e^{\delta\cdot|\mathrm{Im}(z)|}$, is it the case that $U = \hat{u}$, where $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\mathrm{supp}(u) \subset \overline{B_\delta}$?? Yes.

Theorem 4.4: (P-W-S)

(A) If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\mathrm{supp}(\varphi) \subset \overline{B_\delta}$ then $z \mapsto \hat{\varphi}(z)$ is entire and

(+) $|\hat{\varphi}(z)| \lesssim_N (1+|z|)^{-N} e^{\delta|\mathrm{Im}(z)|}$, $z \in \mathbb{C}$, $N = 0, 1, 2, \ldots$

Conversely, if $z \mapsto \Phi(z)$ is entire and satisfies (+) then $\Phi = \hat{\varphi}$ for some $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\mathrm{supp}(\varphi) \subset \overline{B_\delta}$.

(B) If $u \in \mathcal{E}'(\mathbb{R}^n)$ and $\mathrm{supp}(u) \subset \overline{B_\delta}$, then $z \mapsto \hat{u}(z)$ is entire and $\exists N \geq 0$ s.t.

(++) $|\hat{u}(z)| \lesssim (1+|z|)^N e^{\delta\cdot|\mathrm{Im}\, z|}$, $z \in \mathbb{C}^n$.

Conversely, if $z \mapsto U(z)$ is entire and satisfies (++) then $U = \hat{u}$ for some $u \in \mathcal{E}'(\mathbb{R}^n)$ with $\mathrm{supp}(u) \subset \overline{B_\delta}$.

Proof:

Clear that $z \mapsto \hat{\varphi}(z) = \int e^{-iz\cdot x}\varphi(x)\,dx$ is entire [e.g. $\partial\hat{\varphi}/\partial\bar{z}_i = 0$ $i = 1, \ldots, n$ all $z \in \mathbb{C}^n$, or apply Morera + Fubini, or expand $x \mapsto e^{-iz\cdot x}$ and integrate termwise]. For the estimate (+) [c.f. lemma 4.5], for α arbitrary,

$$|z^\alpha \hat{\varphi}(z)| = \left|\int z^\alpha e^{-iz\cdot x}\varphi(x)\,dx\right| = \left|\int e^{-iz\cdot x} D^\alpha \varphi(x)\,dx\right|$$

$$\lesssim_\alpha e^{\delta|\mathrm{Im}\, z|}$$

Since $|e^{-iz\cdot x}| = |e^{\mathrm{Im}\, z\cdot x}| \leq e^{\delta\cdot|\mathrm{Im}\, z|}$ on $\mathrm{supp}\, \varphi$. Estimate (+) now follows.

For converse, given entire $z \mapsto \Phi(z)$ obeying (+), define $\varphi(x) = \frac{1}{(2\pi)^n}\int e^{i\lambda\cdot x}\Phi(\lambda)\,d\lambda$.

Follows ((DCT) + (+)) that $\varphi \in C^\infty(\mathbb{R}^n)$. By Cauchy's theorem, entirety of $z \mapsto \Phi(z)$ and estimate (+), have for arbitrary $\eta \in \mathbb{R}^n$:

$$|\varphi(x)| = \frac{1}{(2\pi)^n}\left|\int e^{i(\lambda+i\eta)\cdot x}\Phi(\lambda+i\eta)\,d\lambda\right|$$

[Justified because of rapid horizontal decay of Φ]. So by (+):

$$|\varphi(x)| \lesssim_N \int e^{-\eta\cdot x}(1+|\lambda+i\eta|)^{-N} e^{\delta|\eta|}\,d\lambda$$

$$\lesssim e^{\delta\cdot|\eta| - \eta\cdot x}$$

Take $\eta = \frac{x}{|x|}t$, $t > 0$.

$$= e^{-t(|x|-\delta)}$$

If $|x| > \delta$, take $t \to \infty$ to get $\varphi = 0$, i.e. $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\mathrm{supp}(\varphi) \subset \overline{B_\delta}$. Taking F.T. shows that $\Phi = \hat{\varphi}$.

(B) ($\Rightarrow$) already established (Lemma 4.7).

($\Leftarrow$) Let $z \mapsto U(z)$ be an entire function satisfying (++). Then $U|_{\mathbb{R}^n} \in \mathcal{S}'(\mathbb{R}^n)$ since $|U(\lambda)| \lesssim \langle\lambda\rangle^N$. Since $\mathcal{F}: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is an isomorphism, $\exists u \in \mathcal{S}'(\mathbb{R}^n)$ s.t. $\hat{u} = U$. Fix $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $\int \varphi\,dx = 1$ and $\mathrm{supp}(\varphi) \subset B_1$. Set $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then $\varphi_\varepsilon \to \delta$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{supp}(\varphi_\varepsilon) \subset B_\varepsilon$. Hence $\hat{\varphi}_\varepsilon \to 1$ in $\mathcal{S}'(\mathbb{R}^n)$. Define $\hat{u}_\varepsilon(z) = \hat{\varphi}_\varepsilon(z)U(z)$. By (+) [for $\hat{\varphi}_\varepsilon$] and (++) [for u], have $|\hat{u}_\varepsilon(z)| \lesssim_N (1+|z|)^{-N} e^{(\varepsilon+\delta)\cdot|\mathrm{Im}(z)|}$, $N = 0, 1, 2, 3, \ldots$ Hence, $u_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and $\mathrm{supp}(u_\varepsilon) \subset \overline{B_{\delta+\varepsilon}}$. As $\varepsilon \downarrow 0$, get $\hat{u}_\varepsilon \to \hat{u}$ in $\mathcal{S}'(\mathbb{R}^n)$

$\square$

LECTURE 13

§5: Oscillatory Integrals:

In this section would like to make sense of

$$\int e^{i\lambda x}\,d\lambda$$

and more generally, objects of the form

$$\int e^{i\Phi(x,\theta)}a(x,\theta)\,d\theta$$

where $x\in X$, $\theta\in\mathbb{R}^k$. Call real valued $\Phi\in C^\infty(X\times\mathbb{R}^k\setminus\{0\})$ the phase function and 'a' will belong to a class of functions called symbols. NOTE: latter integral will not be well-defined classically since we will allow symbols that get larger as $|\theta|\to\infty$.

Lemma 5.1: (Riemann-Lebesgue lemma)
If $f\in L^1(\mathbb{R})$, then $|\hat{f}(\lambda)|\to 0$ as $|\lambda|\to\infty$.

Proof: Assume $f\in L^1(\mathbb{R})\cap C(\mathbb{R})$.

$$\hat{f}(\lambda) = \frac{1}{2}\int\left[e^{-i\lambda x}f(x)+e^{-i\lambda x}f(x)\right]dx$$

$x=x'+\pi/\lambda$

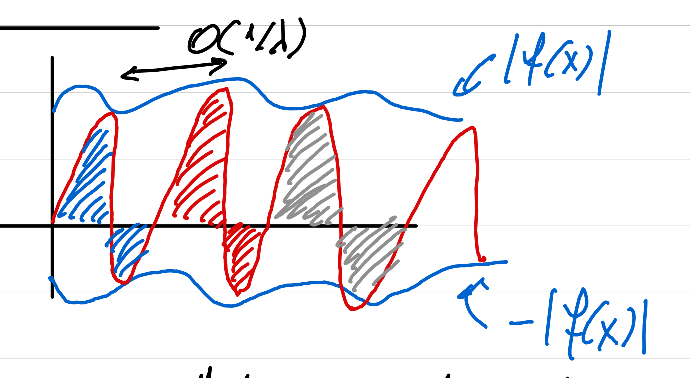
$$= \frac{1}{2}\int e^{-i\lambda x}f(x)+e^{-i\pi}\cdot e^{-i\lambda x}f(x+\tfrac{\pi}{\lambda})\,dx$$

$$= \frac{1}{2}\int e^{-i\lambda x}\cdot\left[f(x)-f(x+\pi/\lambda)\right]dx$$

Since $f\in L^1(\mathbb{R})$, given $\varepsilon>0$, $\exists R$ s.t.

$$\frac{1}{2}\int_{|x|>R}|f(x)-f(x+\pi/\lambda)|\,dx<\varepsilon/4$$

Since $f\in C(\mathbb{R})$, choose $|\lambda|$ sufficiently large so that $\left|\int_{|x|<R}e^{-i\lambda x}\left[f(x)-f(x+\pi/\lambda)\right]dx\right|<\varepsilon/4$ (by DCT). I.e. $|\hat{f}(\lambda)|<\varepsilon/2$ for $|\lambda|$ sufficiently large. Note that $L^1(\mathbb{R})\cap C(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, so given $g\in L^1(\mathbb{R})$, fix $f\in L^1(\mathbb{R})\cap C(\mathbb{R})$ such that $\|f-g\|_{L^1}<\varepsilon/2$. So $|\hat{g}(\lambda)|=|\hat{g}(\lambda)-\hat{f}(\lambda)+\hat{f}(\lambda)|$
$\leq\|g-f\|_{L^1}+|\hat{f}(\lambda)|\leq\varepsilon$ for $|\lambda|$ sufficiently large □.



"More oscillation in integrand ⇒ more decay of integral" because oscillation ⟶ cancellation.

More generally, if $\varphi\in D(\mathbb{R}^n)$ and $\Phi\in C^\infty(\mathbb{R}_n)$ expect

$$\int\varphi(\theta)\cdot e^{i\lambda\Phi(\theta)}\,d\theta$$

to decay as $|\lambda|\to\infty$. E.g. if $\Phi'\neq0$ then the operator $L=\frac{1}{i\lambda\Phi'(\theta)}\frac{d}{d\theta}$ is well-defined since $|\Phi'(\theta)|\gtrsim 1$ on supp φ.
Note that $Le^{i\lambda\Phi}=e^{i\lambda\Phi}$. So

$$\int\varphi(\theta)e^{i\lambda\Phi(\theta)}\,d\theta=\int\varphi(\theta)\,Le^{i\lambda\Phi}\,dx$$
$$=\int L^t[\varphi(\theta)]e^{i\lambda\Phi}\,d\theta$$

where $L^t=-\frac{1}{i\lambda}\frac{d}{d\theta}\left[\frac{1}{\Phi'}\,\bullet\right]$ "formal adjoint of L".

Can do this as many times as we please, so

$$\left|\int e^{i\lambda\Phi}\varphi\,d\theta\right|=\left|\int(L^t)^N(\varphi)e^{i\lambda\Phi}\,d\theta\right|$$
$$\lesssim_N\langle\lambda\rangle^{-N},\quad N=0,1,2,\ldots$$

Expect to get dominant contribution from pts at which $\Phi'=0$ [stationary pts].

Lemma 5.2 (Stationary Phase Lemma): Let $\Phi\in C^\infty(\mathbb{R})$, such that $\Phi'\neq0$ on $\mathbb{R}\setminus\{0\}$ and $\Phi(0)=\Phi'(0)=0$, $\Phi''(0)\neq0$.
Then for $\varphi\in D(\mathbb{R})$:

$$\left|\int e^{i\lambda\Phi(\theta)}\varphi(\theta)\,d\theta\right|\lesssim\frac{1}{|\lambda|^{1/2}},\quad|\lambda|\to\infty.$$

Proof: Fix $\rho\in D(\mathbb{R})$ such that $\rho=1$ on $|\theta|<1$ and $\rho=0$ on $|\theta|\geq2$. Write

$$\int e^{i\lambda\Phi}\varphi(\theta)\,d\theta=\underbrace{\int e^{i\lambda\Phi}\rho(\theta/\delta)\varphi(\theta)\,d\theta}_{I_1}$$
$$+\underbrace{\int e^{i\lambda\Phi}(1-\rho(\theta/\delta))\varphi(\theta)\,d\theta}_{I_2}\qquad\boxed{\delta>0}$$

Since $\rho(\theta/\delta)=0$ on $|\theta|\geq2\delta$, get simple estimate
$|I_1|\lesssim\delta$
Note $(1-\rho(\theta/\delta))=0$ on $|x|<\delta$. So we're integrating over $|x|\geq\delta$, so

$$L=\frac{1}{i\lambda\Phi'}\frac{d}{d\theta}$$

is well-defined and $Le^{i\lambda\Phi}=e^{i\lambda\Phi}$. So

$$I_2(\lambda)=\int e^{i\lambda\Phi}(L^t)^2\left[(1-\rho(\theta/\delta))\cdot\varphi(\theta)\right]d\theta$$

where $L^t=-\frac{1}{i\lambda}\frac{d}{d\theta}\left[\frac{1}{\Phi'}\,\bullet\right]$. Note if

$$P=\frac{d}{d\theta}[a\,\bullet],\quad P^2=\frac{d}{d\theta}\left[a\frac{d}{d\theta}(a\,\bullet)\right]$$
$$=a^2\frac{d^2}{d\theta^2}+3aa'\frac{d}{d\theta}+(aa')'$$

$$(L^t)^2=-\frac{1}{\lambda^2}\left[\frac{1}{(\Phi')^2}\frac{d^2}{d\theta^2}-3\frac{\Phi''}{(\Phi')^3}\frac{d}{d\theta}-\left(\frac{\Phi''}{(\Phi')^3}\right)'\right]$$

$$-\frac{\Phi'''}{(\Phi')^3}+3\frac{(\Phi'')^2}{(\Phi')^4}$$

Note that $\Phi'(\theta)-\cancel{\Phi'(0)}^{\,0}=\int_0^\theta\Phi''(t)\,dt=\theta\int_0^1\Phi''(t\theta)\,dt$
I.e. $\frac{\Phi'(\theta)}{\theta}=\int_0^1\Phi''(t\theta)\,dt$

LHS $\neq0$ at $\theta\neq0$ and $\to\Phi''(0)\neq0$ as $\theta\to0$.
I.e. $|\Phi'(\theta)|\gtrsim|\theta|$ on supp φ.
So

$$(L^t)^2\left[\varphi(\theta)(1-\rho(\theta/\delta))\right]$$
$$=O\left(\frac{1}{\lambda^2\theta^2\delta^2}\right)+O\left(\frac{1}{\lambda^2\theta^3\delta^2}\right)+O\left(\frac{1}{\lambda^2\theta^4}\right)$$

Integrating over $|\theta|>\delta$

$$\Rightarrow|I_2(\lambda)|=O\left(\frac{1}{\lambda^2\delta^3}\right)+O\left(\frac{1}{\lambda^2\delta^3}\right)+O\left(\frac{1}{\lambda^2\delta^3}\right)$$

Matching with $I_1(\lambda)=O(\delta)$, want

$$\delta=\frac{1}{\lambda^2\delta^3}\Rightarrow\delta=\frac{1}{|\lambda|^{1/2}}$$ □

This estimate is sharp, e.g.:

$$\int e^{i\lambda\theta^2}\varphi(\theta)\,d\theta,\quad\theta=\phi/\sqrt{\lambda}$$
$$=\frac{1}{\sqrt{\lambda}}\int e^{i\phi^2}\varphi\left(\frac{\phi}{\sqrt{\lambda}}\right)d\phi\sim\frac{\text{const}}{\sqrt{\lambda}}+\text{lower order terms } |\lambda|\to\infty.$$

Using this, expect

$$u(x)=\int e^{i\Phi(x,\theta)}a(x,\theta)\,d\theta$$

to be "badly behaved" at $x_0\in X$, for which $\nabla_\theta\Phi(x_0,\theta)=0$ for some $\theta\in\mathbb{R}^n$?
We will show that

$$\text{sing supp}(u)\subset\{x\in X:\nabla_\theta\Phi(x,\theta)=0\text{ for some }\theta\in\mathbb{R}^k\setminus\{0\}\}.$$

LECTURE 14

Definition 5.1:

open, $\mathbb{R}^n$

A smooth function $a : X \times \mathbb{R}^k \to \mathbb{C}$ is called a <u>symbol</u> of order $N \in \mathbb{R}$ if:
for each compact $K \subset X$
$|D_x^\alpha D_\theta^\beta a(x,\theta)| \lesssim_{K,\alpha,\beta} \langle \theta \rangle^{N-|\beta|}$ for all $(x,\theta) \in K \times \mathbb{R}^n$. Call space of all such symbols $\mathrm{Sym}(X, \mathbb{R}^k; N)$.

For example if $\{\varphi_\alpha\}$ in $C^\infty(X)$, then
$a(x,\theta) = \sum_{|\alpha| \le N} \varphi_\alpha(x) \cdot \theta^\alpha$ belongs to $\mathrm{Sym}(X, \mathbb{R}^k; N)$.

Only care about behaviour of symbols for large $|\theta|$ since for any compact $L \subset \mathbb{R}^k$, if $a \in C^\infty(X \times \mathbb{R}^k)$ then

$$(x,\theta) \longmapsto \frac{D_x^\alpha D_\theta^\beta a(x,\theta)}{\langle \theta \rangle^{N-|\beta|}}$$

compact in X.

will always be bounded on $K \times L$.

Lemma 5.3:

- If $a \in \mathrm{Sym}(X, \mathbb{R}^k; N) \Rightarrow D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbb{R}^k; N - |\beta|)$
- If $a_i \in \mathrm{Sym}(X, \mathbb{R}^k; N_i)$, $i = 1,2 \Rightarrow a_1 a_2 \in \mathrm{Sym}(X, \mathbb{R}^k; N_1 + N_2)$.

Proof:

Obviously $D_x^\alpha D_\theta^\beta a(x,\theta)$ is smooth on $X \times \mathbb{R}^k$. For $K \subset X$ compact.
$|D_x^{\alpha'} D_\theta^{\beta'} [D_x^\alpha D_\theta^\beta a]| = |D_x^{\alpha'+\alpha} D_\theta^{\beta'+\beta} a|$
$\lesssim_{K,\alpha',\beta'} \langle \theta \rangle^{N-|\beta|-|\beta'|} \Rightarrow D_x^\alpha D_\theta^\beta a \in \mathrm{Sym}(X, \mathbb{R}^k; N - |\beta|)$.

Again for $K \subset X$ compact

$$|D_x^\alpha D_\theta^\beta (a_1 a_2)| = \Big| \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} (D_x^{\alpha'} D_\theta^{\beta'} a_1)(D_x^{\alpha-\alpha'} D_\theta^{\beta-\beta'} a_2) \Big|$$

$$\lesssim_{K,\alpha,\beta} \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} |D_x^{\alpha'} D_\theta^{\beta'} a_1| \cdot |D_x^{\alpha-\alpha'} D_\theta^{\beta-\beta'} a_2|$$

$$\lesssim_{K,\alpha,\beta} \sum_{\alpha' \le \alpha} \sum_{\beta' \le \beta} \langle \theta \rangle^{N_1 - |\beta'|} \langle \theta \rangle^{N_2 - |\beta - \beta'|}$$

$$\lesssim_{K,\alpha,\beta} \langle \theta \rangle^{N_1 + N_2 - |\beta|} \Rightarrow a_1 a_2 \in \mathrm{Sym}(X, \mathbb{R}^k; N_1 + N_2).$$

Lemma 5.4:

If $a \in C^\infty(X \times \mathbb{R}^k)$ and a is positively homogeneous of deg N (in θ) for $|\theta|$ sufficiently large then $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$.

Proof:

For $|\theta|$ sufficiently large $a(x, t\theta) = t^N a(x,\theta)$ for $t > 0$. So for $|\theta|$ large
$t^N D_x^\alpha D_\theta^\beta [a(x,\theta)] = D_x^\alpha D_\theta^\beta [a(x, t\theta)]$
$= t^{|\beta|} (D_x^\alpha D_\theta^\beta a)(x, t\theta)$
i.e. $D_x^\alpha D_\theta^\beta a$ is positively homogeneous of deg $N - |\beta|$, for $|\theta|$ large. For $K \subset X$ compact
$|D_x^\alpha D_\theta^\beta a(x,\theta)| = |D_x^\alpha D_\theta^\beta a(x, |\theta|\omega)|$, $\omega = \frac{\theta}{|\theta|} \in S^{k-1}$.
$= |\theta|^{N-|\beta|} \cdot |D_x^\alpha D_\theta^\beta a(x,\omega)|$
$\lesssim_{K,\alpha,\beta} \langle \theta \rangle^{N-|\beta|}$ $\quad (t = 1/|\theta|, \theta \text{ large})$ $\square$

Definition 5.2:

$\Phi : X \times \mathbb{R}^k \to \mathbb{R}$ is called a <u>phase function</u> if
i) Φ is cts on $X \times \mathbb{R}^k$ and positively homogeneous of deg 1 in θ. $[\Phi(x, t\theta) = t\Phi(x,\theta), t > 0]$.
ii) Φ is smooth on $X \times (\mathbb{R}^k \setminus \{0\})$
iii) $d\Phi = \nabla_\theta \Phi \, d\theta + \nabla_x d\Phi \cdot dx \neq 0$ on $X \times (\mathbb{R}^k \setminus \{0\})$.

Want to make sense of

$$D^\alpha \delta_0(x) = \frac{1}{(2\pi)^n} \int \theta^\alpha e^{ix\cdot\theta} \, d\theta$$

i.e. $\Phi(x,\theta) = x \cdot \theta$, $a(x,\theta) = (2\pi)^{-n} \theta^\alpha \in \mathrm{Sym}(\mathbb{R}^n, \mathbb{R}^n; |\alpha|)$
and more generally, if $x \in X$

$$\int e^{i\Phi(x,\theta)} a(x,\theta) \, d\theta \qquad \to \mathrm{Sym}(X, \mathbb{R}^k; N).$$

$\hookrightarrow$ phase function

<u>Could</u> define a linear form $I_\Phi(a)$: $\mathcal{D}(X) \to \mathbb{C}$ by

$$\langle I_\Phi(a), \varphi \rangle = \int \int e^{i\Phi(x,\theta)} a(x,\theta) \varphi(x) \, dx \, d\theta$$

But is cumbersome because of lack of absolute integrability of the double integral. Instead, fix $\chi \in \mathcal{D}(\mathbb{R}^n)$ s.t. $\chi = 1$ on $|\theta| < 1$ and set

$$I_\Phi^\varepsilon(x) := \int e^{i\Phi(x,\theta)} a(x,\theta) \chi(\varepsilon\theta) \, d\theta.$$

Then define $I_\Phi(a) = \lim_{\varepsilon \downarrow 0} I_\Phi^\varepsilon(a)$ in $\mathcal{D}'(X)$.

Lemma 5.5:

If L has the form

$$L = \sum_{j=1}^{} a_j(x,\theta) \frac{\partial}{\partial \theta_j} + \sum_{j=1}^{n} b_j(x,\theta) \frac{\partial}{\partial x_j} + c(x,\theta)$$

$\in \mathrm{Sym}(X, \mathbb{R}^k; 0)$ $\qquad \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$.

then L^t has same form.

Proof.

$$L^t = -\sum_j \frac{\partial}{\partial \theta_j}(a_j \cdot) - \sum_j \frac{\partial}{\partial x_j}(b_j \cdot) + c$$

$$= \sum \tilde{a}_j \, \partial/\partial\theta_j + \sum \tilde{b}_j \, \partial/\partial x_j + \tilde{c}$$

$-a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$ $\qquad -b_j \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$

$$- \sum \frac{\partial a_j}{\partial \theta_j} - \sum \frac{\partial b_j}{\partial x_j} + c$$

$\in \mathrm{Sym}(X, \mathbb{R}^k; -1)$ (use lemma).

If we could find such an L for which $Le^{i\Phi} = e^{i\Phi}$ then

$$\langle \Phi_I^\varepsilon(a), \varphi \rangle = \iint (L^N e^{i\Phi}) a(x,\theta) \chi(\varepsilon\theta) \varphi(x) \, dx \, d\theta$$

$$= \iint e^{i\Phi} (L^t)^N [a(x,\theta) \chi(\varepsilon\theta) \varphi(x)] \, dx \, d\theta.$$

form of L, (L^t) should <u>lower</u> order of $[a(x,\theta) \chi(\varepsilon\theta) \varphi(x)]$ by 1 each time.

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Lemma 5.5:
If
$$L = \sum_{j=1}^{k} a_j \frac{\partial}{\partial \theta_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c$$
$a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$, $b_j, c \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$

then L^t has same form

Lemma 5.6:
There exists a differential operator L of the form:
$$L = (+)$$
such that $L^t e^{i\Phi} = e^{i\Phi}$, where Φ is any (fixed) phase function.

Proof:
Clearly, $\frac{\partial}{\partial \theta_j} e^{i\Phi} = i \frac{\partial \Phi}{\partial \theta_j} e^{i\Phi}$, $\frac{\partial}{\partial x_j} e^{i\Phi} = i \frac{\partial \Phi}{\partial x_j} e^{i\Phi}$

So
$$\left(\sum_{j=1}^{k} -i|\theta|^2 \frac{\partial \Phi}{\partial \theta_j} \frac{\partial}{\partial \theta_j} + \sum_{j=1}^{n} -i \frac{\partial \Phi}{\partial x_j} \frac{\partial}{\partial x_j} \right) e^{i\Phi}$$
$$= \left(|\theta|^2 |\nabla_\theta \Phi|^2 + |\nabla_x \Phi|^2 \right) e^{i\Phi}$$

Note, since $\Phi(x, t\theta) = t\Phi(x, \theta)$, $t > 0$
$$t \frac{\partial}{\partial x_j} \Phi(x, \theta) = \frac{\partial}{\partial x_j} \Phi(x, t\theta) = \frac{\partial \Phi}{\partial x_j}(x, t\theta)$$
So $\partial\Phi / \partial x_j$ +vley homogeneous of deg 1.
$$t \frac{\partial}{\partial \theta_j} \Phi(x, \theta) = \frac{\partial \Phi}{\partial \theta_j}(x, t\theta) = t \frac{\partial \Phi}{\partial \theta_j}(x, t\theta)$$
So $\partial \Phi / \partial \theta$ is +vley homogeneous of deg. 0.

Define
$$P = \sum_{j=1}^{k} \tilde{a}_j \, \partial / \partial \theta_j + \sum_{j=1}^{n} \tilde{b}_j \, \partial / \partial x_j$$
$$\tilde{a}_j = \frac{-i|\theta|^2 \partial \Phi / \partial \theta_j}{|\theta|^2 |\nabla_\theta \Phi|^2 + |\nabla_x \Phi|^2} \qquad \tilde{b}_j = \frac{-i \partial \Phi / \partial x_j}{|\theta|^2 |\nabla_\theta \Phi|^2 + |\nabla_x \Phi|^2}$$
So $P e^{i\Phi} = e^{i\Phi}$

See that $\tilde{a}_j$ +vley homogeneous of deg 0,
—"— $\tilde{b}_j$ —"— deg −1.
Note that denominators can vanish at $\theta = 0$.
Fix $\rho \in D(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$. Define
$$L^t = (1-\rho) P + \rho$$

Then $L^t e^{i\Phi} = (1-\rho) e^{i\Phi} + \rho e^{i\Phi} = e^{i\Phi}$, by lemma 5.5 + 5.4
$$L = \sum_{j=1}^{k} a_j \frac{\partial}{\partial \theta_j} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c. \quad (:= (L^t)^t)$$
$a_j \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$, $b_j, c \in \mathrm{Sym}(X, \mathbb{R}^k; -1)$.

Note that
$$L : \mathrm{Sym}(X, \mathbb{R}^k; N) \to \mathrm{Sym}(X, \mathbb{R}^k; N-1).$$

Also, more generally
$$L^M [a(x,\theta) \varphi(x)] = \sum_{|\alpha| \le M} a_\alpha(x, \theta) \partial^\alpha \varphi$$
$a_\alpha \in \mathrm{Sym}(X, \mathbb{R}^k; N-M)$

[induction on M].

Theorem 5.1:
If Φ is a phase function and $a \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ then
$$I_\Phi(a) = \lim_{\varepsilon \downarrow 0} I_\Phi^\varepsilon(a) \in D'(X)$$
and $\mathrm{ord}(I_\Phi(a)) \le N + k + 1$.

Proof:
For each $\varepsilon > 0$
$$I_\Phi^\varepsilon(a) = \int e^{i\Phi(x,\theta)} a(x, \theta) \chi(\varepsilon\theta) \, d\theta$$
$\chi \in D(\mathbb{R}^k)$, $\chi = 1$ on $|\theta| < 1$, $= 0$ on $|\theta| > 2$

So for $\varphi \in D(X)$:
$$\langle I_\Phi^\varepsilon(a), \varphi \rangle = \iint e^{i\Phi(x,\theta)} a(x,\theta) \chi(\varepsilon\theta) \varphi(x) \, dx \, d\theta$$
$$= \iint [(L^t)^M e^{i\Phi}] a(x,\theta) \chi(\varepsilon\theta) \varphi(x) \, dx \, d\theta$$
L as in lemma 5.6
$$= \iint e^{i\Phi} L^M [a(x,\theta) \chi(\varepsilon\theta) \varphi(x)] \, dx \, d\theta$$
Note that, since $\chi \in D(\mathbb{R}^k)$
$$\left| \left(\frac{\partial}{\partial \theta} \right)^\alpha \chi(\varepsilon\theta) \right| = \varepsilon^{|\alpha|} |(\partial^\alpha \chi)(\varepsilon\theta)| \lesssim_\alpha \varepsilon^{|\alpha|} \langle \varepsilon\theta \rangle^{-|\alpha|}$$
$$= C_\alpha \cdot \frac{\varepsilon^{|\alpha|}}{[1 + \varepsilon^2 |\theta|^2]^{|\alpha|/2}} = C_\alpha \cdot \frac{1}{[1/\varepsilon^2 + |\theta|^2]^{|\alpha|/2}}$$
So, for $0 < \varepsilon \le 1$:
$$|(\partial/\partial\theta)^\alpha \chi(\varepsilon\theta)| \lesssim_\alpha \langle \theta \rangle^{-|\alpha|}$$
i.e. $\chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^k; 0)$ uniformly in ε.
So $a(x,\theta)\chi(\varepsilon\theta) \in \mathrm{Sym}(X, \mathbb{R}^k; N)$ so
$$L^M [a(x,\theta) \chi(\varepsilon\theta) \varphi(x)] = \sum_{|\alpha| \le M} a_\alpha(x, \theta; \varepsilon) \partial^\alpha \varphi$$
$a_\alpha(x,\theta;\varepsilon) \in \mathrm{Sym}(X, \mathbb{R}^k; N-M)$.

And also $a_\alpha(x,\theta) := a_\alpha(x, \theta; 0) \in \mathrm{Sym}(X, \mathbb{R}^k; N-M)$.
Choose M sufficiently large, i.e. $N - M \le -(k+1)$ i.e. enough to take $M = N + k + 1$.
So, by DCT
$$\langle I_\Phi(a), \varphi \rangle = \lim_{\varepsilon \downarrow 0} \langle I_\Phi^\varepsilon(a), \varphi \rangle$$
$$= \sum_{|\alpha| \le N+k+1} \iint e^{i\Phi(x,\theta)} a_\alpha(x,\theta) \partial^\alpha \varphi_m \, dx \, d\theta$$

If $\mathrm{supp}(\varphi) \subset K$, then
$$|\langle I_\Phi(a), \varphi \rangle| \le \sum_{|\alpha| \le N+k+1} \iint |a_\alpha(x,\theta)| \cdot |\partial^\alpha \varphi| \cdot dx \, d\theta$$
$$\lesssim_K \sum_{|\alpha| \le N+k+1} \sup |\partial^\alpha \varphi| \quad (a_\alpha\text{'s integrable})$$

so $I_\Phi(a) \in D'(X)$ and $\mathrm{ord}(I_\Phi(a)) \le N + k + 1$ □

Given $I_\Phi(a) \in D'(X)$
$$\int e^{i\Phi(x,\theta)} a(x,\theta) \, d\theta$$

Can show that $\partial/\partial x_j I_\Phi(a)$ coincides with oscillatory integral
$$\int e^{i\Phi(x,\theta)} \left[i \, \partial\Phi/\partial x_j \, a(x,\theta) + \frac{\partial a}{\partial x_j}(x,\theta) \right] d\theta$$

* Since $[\cdots]$ might fail to be smooth at $\theta = 0$, write
$$\int e^{i\Phi(x,\theta)} \rho(\theta) a(x,\theta) \, d\theta + \int e^{i\Phi(x,\theta)} (1-\rho)(\theta) a(x,\theta) \, d\theta$$

where $\rho \in D(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$.
Because of this technicality, often assume that support of $a(x,\theta)$ lies in $|\theta| > 1$.

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Consider

$$I_{\Phi}(a) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\theta}\, d\theta, \quad (x,\theta) \in \underset{X}{\mathbb{R}^n} \times \underset{\mathbb{R}^k}{\mathbb{R}^n}$$

For $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\langle I_{\Phi}(a), \varphi \rangle \boxed{=} \lim_{\varepsilon \downarrow 0} \iint e^{ix\cdot\theta} \chi(\varepsilon\theta)\varphi(x)\, dx\, d\theta$$

$(x,\theta) \mapsto (\varepsilon x, \theta/\varepsilon)$

$$\boxed{=} \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^n} \iint e^{ix\cdot\theta} \chi(\theta)\varphi(\varepsilon x)\, dx\, d\theta$$

$$= \lim_{\varepsilon\to 0} \int \frac{1}{(2\pi)^n} \hat{\chi}(-x)\varphi(\varepsilon x)\, dx$$

$$= \varphi(0) \cdot \int \frac{1}{(2\pi)^n} \hat{\chi}(-x)\, dx = \varphi(0)\cdot \overset{=1}{\chi(0)} = \varphi(0)$$

I.e. $\delta_0(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\theta}\, d\theta$

This gives

$$D_0^{\alpha}\delta(x) = \frac{1}{(2\pi)^n} \int \theta^{\alpha} e^{ix\cdot\theta}\, d\theta$$

Natural to ask when $I_{\Phi}(a) \in \mathcal{D}'(X)$ can be identified with a smooth function.

Def 5.3:
Let $Y \subset X$ be open. Say $u \in \mathcal{D}'(X)$ is smooth on Y if there exists $f \in C^{\infty}(Y)$ such that $\langle u, \varphi\rangle = \int f\varphi\, dx$ for all $\varphi \in \mathcal{D}(Y)$. Define singular support of $u \in \mathcal{D}'(X)$ by:

$$\text{sing supp}(u) = X \setminus \bigcup_{\substack{Y \subset X \\ \text{open}}} \{Y : u \text{ is smooth on } Y\}$$

[I.e. complement of largest open set on which u is smooth.]

E.g. $\text{sing supp}(\delta_0) = \{0\}$.

When looking at sing supp of $I_{\Phi}(a)$ following lemma allows us to assume $a(x,\theta) = 0$ on $|\theta| < 1$ WLOG.

Lemma 5.7:
If Φ is a phase function, a symbol then the function $x \longmapsto \int e^{i\Phi(x,\theta)}\rho(\theta)a(x,\theta)\, d\theta$ is smooth for any $\rho \in \mathcal{D}(\mathbb{R}^k)$.

Fix $\rho \in \mathcal{D}(\mathbb{R}^k)$, $\rho = 1$ on $|\theta| < 1$ and $\rho = 0$ on $|\theta| > 2$, can write

$$I_{\Phi}(a) = \underbrace{I_{\Phi}(\rho a)}_{\text{smooth } f^n} + I_{\Phi}(\underbrace{(1-\rho)a}_{=\tilde{a}})$$

Note $a \in \text{Sym}(X, \mathbb{R}^k; N) \Rightarrow \tilde{a} \in \text{Sym}(X, \mathbb{R}^k; N)$ and $\text{sing supp}\, I_{\Phi}(a) = \text{sing supp}\, I_{\Phi}(\tilde{a})$
Clearly $\tilde{a}(x,\theta) = 0$ on $|\theta| < 1$.

$$\int e^{i\Phi(x,\theta)} a(x,\theta)\, d\theta$$

Expect this to be "bad" at $x \in X$ for which $\nabla_{\theta}\Phi(x,\theta) = 0$ for some $\theta \in \mathbb{R}^k$.

Theorem 5.2:
$\text{sing sup}\, I_{\Phi}(a) \subset \{x \in X : \nabla_{\theta}\Phi(x,\theta) = 0$ for some $\theta \in \mathbb{R}^k \setminus \{0\}\}$.

Proof:
Fix $x_0 \in X$ for which $\nabla_{\theta}\Phi(x,\theta) \neq 0$ $\forall \theta \in \mathbb{R}^k \setminus \{0\}$.
Note $\theta \longmapsto |\nabla_{\theta}\Phi(x_0,\theta)|$ is homogeneous of deg 0, so is completely determined by values it takes on S^{k-1}. By continuity and compactness

$$|\nabla_{\theta}\Phi(x_0,\theta)| \gtrsim 1 \text{ on } \mathbb{R}^k \setminus \{0\}$$

By continuity, $\exists$ small open nbhd Y of x_0 such that $|\nabla_{\theta}\Phi(x,\theta)| \gtrsim 1$ on $Y \times (\mathbb{R}^k \setminus \{0\})$.

Consider

$$\varphi \longmapsto \langle I_{\Phi}(a), \varphi\rangle, \quad \varphi \in \mathcal{D}(Y).$$

The differential operator

$$L^t = \sum_{j=1}^{k} \frac{-i\,\partial\Phi/\partial\theta_j}{|\nabla_{\theta}\Phi|} \cdot \frac{\partial}{\partial\theta_j}$$

Well-defined on $Y \times (\mathbb{R}^k \setminus \{0\})$, and $L^t e^{i\Phi} = e^{i\Phi}$. Since we can assume $a(x,\theta) = 0$ on $|\theta| < 1$ WLOG, follows that L^t is well-defined on

$$(Y \times \mathbb{R}^k) \cap \text{supp}[a(x,\theta)].$$

By same argument as proof of Th^m 5.1,

$$L : \text{Sym}(X, \mathbb{R}^k; N) \longrightarrow \text{Sym}(X, \mathbb{R}^k; N-1).$$

So for $\varphi \in \mathcal{D}(Y)$

$$\langle I_{\Phi}(a), \varphi\rangle = \lim_{\varepsilon\downarrow 0} \iint e^{i\Phi} L^M[a(x,\theta)\chi(\varepsilon\theta)]\varphi(x)\, dx\, d\theta$$

Note: L does not hit $\varphi(x)$. Since $a(x,\theta)\chi(\varepsilon\theta) \in \text{Sym}(X, \mathbb{R}^k; N)$ (uniformly in ε) can choose M large enough so we use DCT to take limit:

$$\langle I_{\Phi}(a), \varphi\rangle = \int \Big[\underbrace{\int e^{i\Phi} L^M a(x,\theta)\, d\theta}_{(\dagger)}\Big]\varphi(x)\, dx$$

Can choose M as large as we please, so differentiation under integral permitted and deduce $I_{\Phi}(a)$ is smooth on Y.
Since $x_0 \in Y \Rightarrow$ result $\square$

$$\delta_0(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\theta}\, d\theta$$

$$\text{sing supp}(\delta_0) \subset \{x \in \mathbb{R}^n : \nabla_{\theta}(x\cdot\theta) = 0 \text{ some } \theta \in \mathbb{R}^k \setminus \{0\}\} = \{x = 0\}$$

Suppose want to solve

$$\frac{\partial u}{\partial t} + \underline{c}\cdot\nabla u = 0, \quad \lim_{t\downarrow 0} u(x,t) = \delta_0(x)$$

I.e. $u(\cdot,t) \in \mathcal{D}'(\mathbb{R}^n)$ $\forall t$ and $\lim_{t\downarrow 0} u(\cdot,t) = \delta_0(x)$

Set $X = (\underline{x}, t)$. Guess, by F.T.

$$u(x,t) = \frac{1}{(2\pi)^n} \int e^{i\theta\cdot(x - \underline{c}\cdot t)}\, d\theta$$

Differentiate under integral, find

$$\frac{\partial u}{\partial t} + \underline{c}\cdot\nabla u = 0.$$

and $\lim_{t\downarrow 0} u(\underline{x}, t) = \frac{1}{(2\pi)^n} \int e^{i\theta\cdot x}\, d\theta = \delta_0(x)$ in $\mathcal{D}'(\mathbb{R}^n)$.